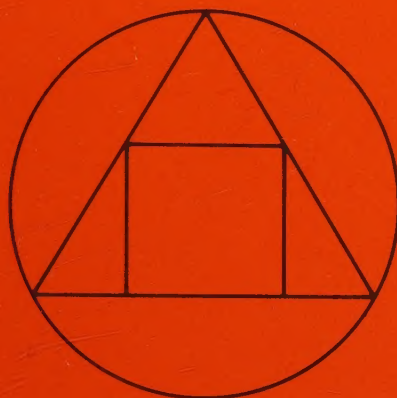


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Lund Technical University
Lund , Sweden



On Hydrodynamic Torque Converters

Sven Andersson

Lund 1982

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ON HYDRODYNAMIC TORQUE CONVERTERS

Av

Sven Andersson

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Abstract Analysis of hydrodynamic torque converters is the subject of this report. The flow is assumed to be two-dimensional, that is no variation in the circumferential direction, frictionless, and stationary. Infinitely thin blades and constant fluid density are also assumed. Governing equations and applicable numerical techniques are examined and numerical results are obtained for the three-element torque converter W 240 (Fichtel & Sachs). The torques thus obtained appeared to be only slightly greater, at most 10% for the pump and 6% for the turbine, than the torques obtained from the frequently used one-dimensional theory. Some experiments were carried out with the actual torque converter. Measured and calculated turbine torques were compared and it was found that the two-dimensional theory produced satisfactory values, a little better than the one-dimensional one. A friction factor which accounts for the dissipation of real flow was also evaluated. An exploration of other modes of operation, including backwards rotation of pump and turbine, showed that satisfactory prediction of these is possible in some cases.			
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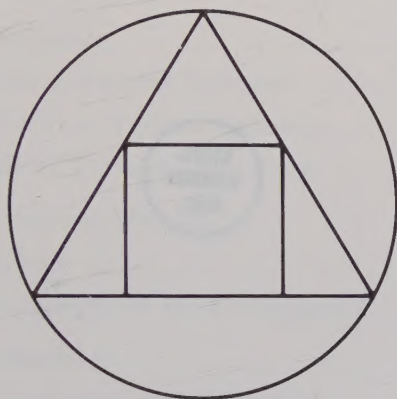
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Sven Andersson



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1. Introduction

The hydrodynamic torque converter, which was invented in 1905 by *Hermann Föttinger*[†], consists basically of a set of turbines arranged in a common casing and forming a closed flow circuit of toroidal shape.

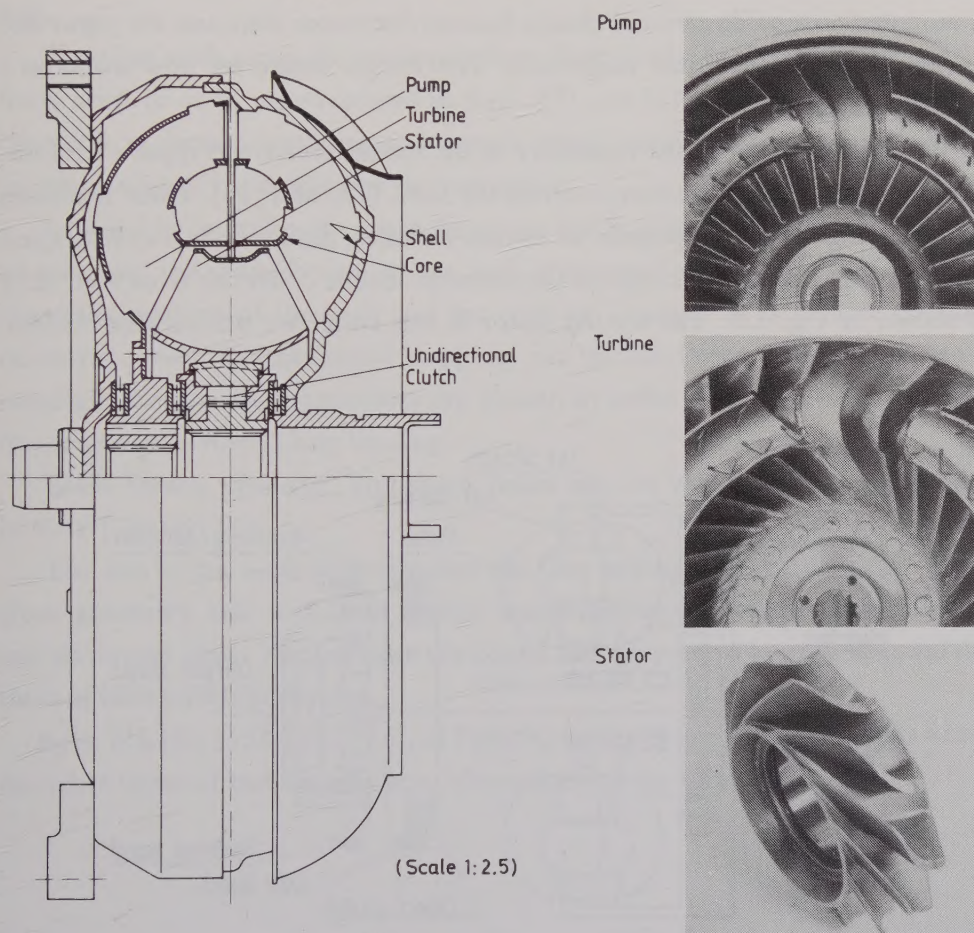


Fig. 1.1. Torque converter W240, (Fichtel & Sachs AG, Schweinfurt, BRD)

[†] *Hermann Föttinger* (1877-1945), german engineer. Refs. [8], [23], and [26] present some historical facts.

The most frequent use of torque converters is in automatic transmissions for automobiles. Fig. 1.1 shows such a converter and also introduces some terminology.

The pump is connected to the driving engine, the turbine drives the output shaft, and the stator is held stationary serving as reaction member. When the output shaft is stopped the output torque is usually about $2 \div 3$ times the input torque. The output torque decreases as the speed of the output shaft increases and has the same magnitude as the input torque when the output speed is about $80 \div 90\%$ of the input speed. A further decrease of the output torque would imply a reversal of the reaction torque, which however is prevented as the stator is mounted on a unidirectional clutch. Instead the stator idles and the input and output torques have equal magnitude. The torque converter now works as a fluid coupling.

In order to improve the capability of the torque converter, types with four, five, or six elements have been evolved, see Refs. [23] and [26]. These machines are often ingenious compounds of various turbines, unidirectional clutches, and epicyclical gears. As an example a five-element torque converter from Ref. [23] is shown in Fig. 1.2. The second stator in this converter has turnable blades,

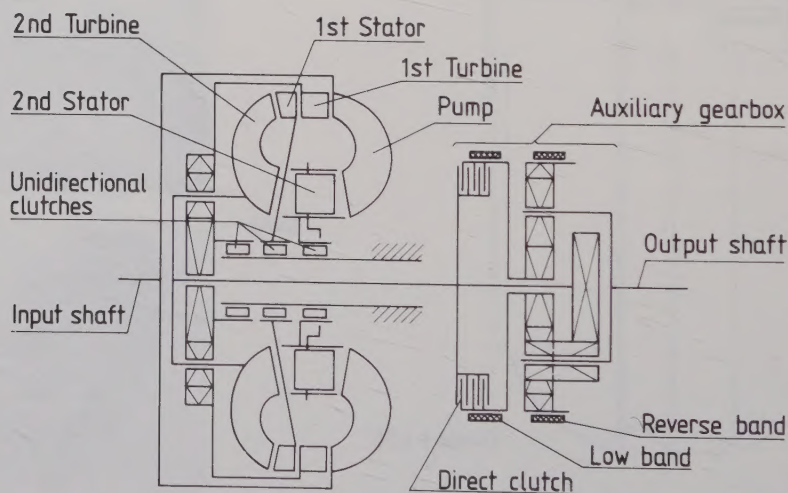


Fig. 1.2. Scheme of the Buick Twin-Turbine transmission

which provides a means of direct control. The auxiliary gearbox provides a re-

verse gear and a low forward gear, but during normal driving only the direct clutch is used and hence no manoeuvres of clutches and brake bands are required as for ordinary automatic transmissions.

The analysis of torque converters is usually carried out as if the flow was concentrated to a middle line in the flow circuit. This method is referred to as the one-dimensional method and is treated by Refs. [4], [8], [19], [24], [26], and [27]. The essence of the method is a power balance for the flow circuit. In Ref. [24] this power balance is evaluated in a more precise way by proposing certain variations of various quantities over the widths of the turbines. This method is reported to improve the agreement between calculated and measured data.

Treatises with a two-dimensional effect, that is varying flow over the width, from shell to core, are presented in Refs. [7] and [12]. These have an inverse character as certain assumptions are made about certain flow quantities leading to calculation of the blading of the turbines.

In Ref. [12] a choice of certain functions yields a flow pattern and the shape of the core; the cross section of the shell is assumed to be a circle. In Ref. [7] the cross sections of the shell and the core are assumed to be almost circular eccentric curves. By conformal mappings the enclosed area is transformed to a rectangle. Certain flow quantities are chosen in order to achieve an acceptable flow and a prescribed blade loading.

Blade forces, pressures, and shock losses are not very explicitly dealt with in Refs. [7] and [12].

The aim of this work is to calculate the flow and associated quantities from given geometry and rotational speeds assuming two-dimensional, frictionless, and stationary flow. Furthermore the blades are assumed to be very thin and the fluid to have constant density.

Refs. [1], [9], [13], [14], [15], [17], [21], and [22] are treatises of this kind for other types of turbomachinery, often steam or gas turbines.

2. Notation

For most symbols, reference to a chapter or a section is given in order to provide a precise definition.

A matrix is denoted by a letter of the type: $\mathbf{A}$, whilst an element in its i 'th row and j 'th column is denoted a_{ij} .

Vectors and column matrices are denoted by an arrow, $\vec{a}$.

A, B, C, D	Corner points, 3.1
A, B	Points just in front of and behind an entrance shock respectively, 4.1
A	Area, 3.2
A, B, C	Constants for Ω , 7.1
$\vec{A}$	Area vector, 3.2
$A_{0,\text{eff}}$	Effective through flow area, 7.4
A, B, C	Functions in equation for c_m , 3.4
A, B	General functions, 3.4
a	Area coefficient, 7.1
$\hat{a}$	Unit vector perpendicular to k-line, 3.1
a_0	Effective area coefficient, 7.4
a, b, c	Constants in N , 6.3
a, b	Fourier coefficients, 4.3
a, B, C, k, n	Constants, 8.1
a, b, c, d, e, f, g, u	General functions, 3.4
B	Blade surface function, 3.1
b	Width, 4.3, 5.2
b	Band width, 6.3
C, D, P, Q, T	Points, 3.1
C, D, E, G, H, J	Points, 4.1
C, D	Boundary curve, domain, 6.1
$\vec{C}$	Vector, 7.4

$\vec{c}$	Fluid velocity, 3.1
D, H, K, L, R	Matrices used in the FEM, 6.1, 6.3
D, G, Q, T	Matrices for a transmission, 7.4
D_h	Hydraulic diameter, 7.1
$\vec{f}$	Force per unit mass parallel to the blades, 3.3
d	Diameter, 10
E	Point at blade exit, 4.1
E	Power, 3.5
$\vec{e}$	Vector, 4.2
e	Fourier coefficient, 5.2
e_D	Constant due to numerical method, 5.2
e	Residue, 6.1
F, F_{ax}	Axial thrust force, 3.5
$\vec{f}$	Force per unit mass, 3.3
f	Friction factor, 10
f	Quantity, 5.1
g	Quantity, 6.1
H	Rothalpy, 3.3
h	Distance between k-lines, 5.2
h	Fourier coefficient, 7.3
h	Height, 4.1
i, k	Curvilinear coordinates, 3.1
I	Specific energy, 3.3
ΔI	Specific energy loss, 4.1, 7.1
$I_{1/r}$	An integral, 6.3
J_z	Momentum at a k-line, 3.5
K	Curvature, 3.2
$\vec{K}$	Blade force, 4.1
$\vec{k}$	Force per unit mass perpendicular to the blades, 3.3
$\vec{L}$	Moment of momentum, 3.5
$\hat{l}$	Unit vector in the meridional plane, 3.1
l	Length of pipe, 7.1
M	Torque, 3.5
M_d	Drag torque, 10
M	Base function, 6.1

m, n	Number of nodes and triangles respectively, 6.1, 6.3
m	Length of streamline, 3.1
$\dot{m}$	Mass flow, 3.2
m	Mass, 3.2
N	Number of turbines, 7.1
N_ω	Number of equations for angular velocities, 7.4
N	Local base function, 6.3
N	General function, 3.1
$\vec{n}$	Vector perpendicular to the blades, 3.1
n	Rotational speed in rpm, 10
n	Iteration cycle number, 7.3
O, S	Points, 3.1
P, Q	Points, 7.1
$\vec{P}$	Momentum, 3.3
p	Pressure, 3.3
$\vec{Q}$	Lagranges interpolation polynom, 5.1
$\vec{Q}$	Pressure force, 4.2
q, q_x	Quantities, 6.1
q	General function, 5.1
Re, Re_D	Reynolds numbers, 10
R_D	Outer radius, 3.6
R_K	Radius of curvature, 3.2
r, θ, z	Cylinder coordinates, 3.1
$\vec{r}$	Radius vector, 5.1
S, Y	Constants, 7.1
s, v	Length of i, k-lines, 3.1
s	Length coordinate along boundary curve, 6.1
T	Number of traced streamlines, 5.1
T_e, T_C, T_b	Sets containing the node numbers for: the e 'th triangle, the boundaries, the nodes where Ψ is assigned, 6.3
t	Time, 3.2
U_k	Set containing the triangle numbers around node k , 6.3
$\vec{u}$	Tangential velocity of turbine blades, 3.1
$\dot{V}$	Volume flow, 3.2
V	Volume, 3.2

v_S	Shock velocity, 4.1
W	Power flux, 3.5
ΔW	Power loss, 7.1
$\vec{w}$	Relative fluid velocity, 3.1
w_{ABn}	Component of $\vec{w}_{AB}$ perpendicular to the blades, 4.2
X, x	General quantities, 3.2
x, y, z	Cartesian coordinates, 3.1
$\vec{x}, \vec{y}$	Vectors, 6.3
Z	Number of teeth, 7.4
α, β	Angle between $\hat{z}$ and k-line, and $\hat{z}$ and i-line respectively, 3.1
α	Blade angle coefficient, 7.1
$\vec{B}, \vec{\beta}$	Vectors, 7.4
Γ	Specific moment of momentum, 3.3
γ	Angle between c_z and c_m , 3.1
$\Delta ()$	Increment of change of the quantity ()
δ	Deviation angle, 4.1
δ	Displacement of streamline position, 7.1
ϵ	Corrective velocity, 5.1
$\vec{\zeta}$	Vorticity, 3.3
$\zeta_{\theta, F}, \zeta_{\theta, \Psi}$	Versions of ζ_{θ} , 3.4
η	Efficiency, 9
η, σ	Functions, 6.2
Θ	Blade surface angle, 3.1
ϑ	Oil temperature, 10
ι	Fourier coefficient, 4.3
κ	Shock factor, 4.1
Λ	Damping coefficient for stream function, 5.1, 6.3
Λ_H	Damping coefficient for rothalpy, 7.2
Λ_{ω}	Damping coefficient for angular velocities, 7.4
λ, λ_v	Angle between $\hat{a}$ and c_m , and $\hat{a}$ and i-line respectively, 3.1, 4.2
λ_D	Fluid friction coefficient, 7.1
μ	Torque multiplication, 9
ν	Dynamic viscosity, 10

ν	Direction constant, 7.4
Ξ	Functional, 6.2
ξ	Constant, 7.3
ρ	Fluid density, 3.2
σ	Relative width, 8.1
τ	Angle between $\hat{z}$ and $\hat{l}$, 3.1
τ	Parameter, 5.1
τ	Local coordinate, 5.2
Υ, χ	Operators, 3.4
v	General function, 6.1
Φ	Dissipation function, 3.3
$\phi_{\odot}$	Blading factor, 10
φ	Blade angle, 3.1
Ψ, ψ	Stream function, stream function value, 3.2, 6.1
Ψ^{-1}	Inverse function for Ψ , 5.1
Ψ_{ϵ}	Corrective stream function, 5.1
Ω	Quantity, 7.1
ω	Angular velocity, 3.1
ω_p	Characteristic value for angular velocities, 3.6

Subscripts

A, B	Just in front of and behind an entrance shock respectively, 4.1
A, B, C, D, E	Shafts, 7.4
AB, AC, BD, CD, CA, DC	Parts of the boundary curve, 3.1, 3.5, 6.1
AB	Just in front of an entrance shock in the horizon of B, 4.1
$a, l, m, r, \theta, s, v, x, y, z$	In the corresponding directions, 3.1
approx, exact, max	Approximative, exact, and maximal value respectively, 8.1
BA	At B but with c_{mB} exchanged for c_{mA} , 4.2
C	Correct value, 5.2
core, shell	At the core and shell respectively, 1, 7.1
D	Due to dissipation, 3.5
e, i, j, k, l, p	Integers
inlet	Inlet to a row of turbines, 8.2
i, k	At an i- and k-line respectively, 3.1

o	Optimal value, 5.2
P, T	Pump and turbine, 10
P, Q	At points P and Q, 7.1
R	Renewed value, 6.3
S	Due to entrance shock, 4.1
S	At point S, 3.1
steady	Steady state value, 7.1
t	At time t , 3.2
V	For volume V , 3.2
x, y	Derivations with respect to x and y respectively, 3.4
α, ψ	For a specified α and ψ value respectively, 8.1
0	Non-dimensional value, 3.6
0	For a special k-line, 7.1
0	At $z = 0$, 4.3
∞	When some quantity is infinite, 4.3

Superscripts

(e)	For the e 'th triangle, 6.3
$(n), (\mu)$	Iteration cycle numbers, 5.1, 6.3, 7.1
1	According to one-dimensional theory, 9
$'$	Derivative or similar point
$''$	Second derivative or yet another similar point
$*$	With respect to a mean entrance direction, 5.1
$\wedge$	Unit vector or amplitude
$\sim$	Perturbed quantity, 7.1
$—$	Mean value, 7.4

3. The Flow Through a Turbine

3.1. Geometry and Coordinate Systems

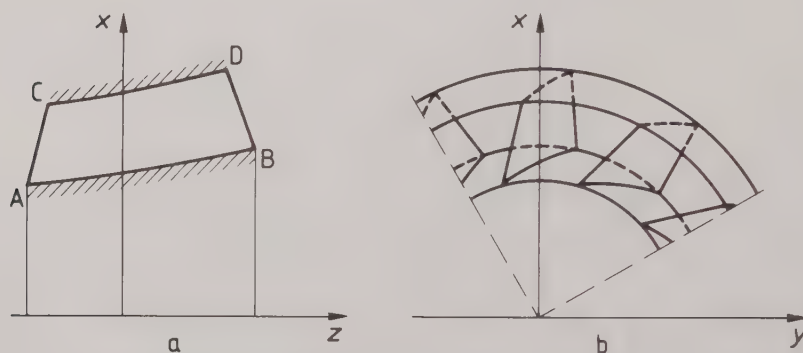


Fig. 3.1. Cross section a) and front view b) of a turbine

The first step in the study of hydrodynamic torque converters is to investigate the flow through a single turbine. In Fig. 3.1.b such a machine is shown in front view while Fig. 3.1.a shows its cross section. The four lines AC, BD, AB, and CD generate four surfaces of revolution which enclose the space under consideration. Fluid is assumed to enter this space at the AC surface and to leave it at the BD surface. These two surfaces are assumed to be conical, that is the AC and BD lines are straight. The AB and CD surfaces are walls which form parts of the shell and the core, respectively, of the torque converter. It is obvious that the use of cylinder coordinates (r, θ, z) , see Fig. 3.2, is convenient.

The energy transfer between the fluid and the turbine is brought about by a number of blades in the ACDB space. The fluid is exposed to a concentrated force at each blade as there are different pressures on its two sides; the blade has a suction side and a pressure side. If the turbine has many blades, which is usually the case in torque converters, these discrete forces may be replaced by a continuous field of forces. This is equivalent to considering a turbine with an

infinite number of blades. In this fictive turbine the flow is independent of the circumferential coordinate θ . The real three-dimensional flow is thus replaced by a two-dimensional one which is a mean quantity with respect to the θ coordinate of the real flow. This concept is widely used and Refs. [1], [9], [10], [12], [13], [14], [15], [17], [21], and [22] show the use and validity of it in the analysis of turbomachinery.

When a fluid particle passes through the turbine it follows a certain streamline. Figs. 3.2.a and 3.2.b show the projections of this three-dimensional curve on the $x - y$ plane and the $r - z$ plane respectively. The $r - z$ plane is also referred to as the meridional plane and the corresponding projection of the streamline as the meridional streamline.

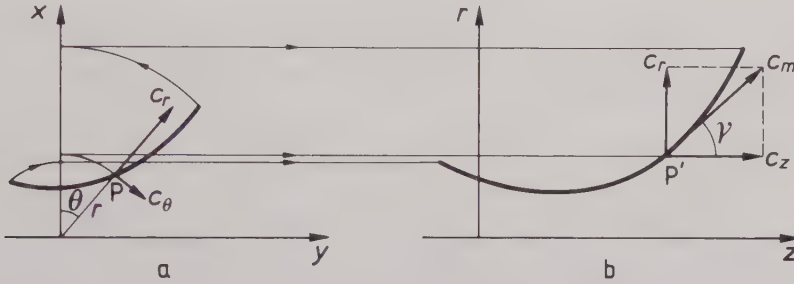


Fig. 3.2. A streamline in the $x - y$ plane view a), and the corresponding meridional streamline b)

The velocity of the fluid at a certain point P , see Fig. 3.2, is named $\vec{c}$. This may be resolved in two components, c_θ in the circumferential direction and c_m which is contained in the meridional plane. The meridional velocity c_m may then be resolved in c_r and c_z which are the velocities in the radial and the axial directions respectively. The angle between c_m and c_z is denoted γ , giving

$$\begin{cases} c_z = c_m \cos \gamma \\ c_r = c_m \sin \gamma \end{cases} \quad (3.1)$$

The velocity of the fluid may also be resolved in the two components $\vec{w}$, which is the velocity relative to the turbine, and $\vec{u}$ due to the rotation of the turbine. Fig. 3.3, which is an extension of Fig. 3.2.b, shows the different velocities. The angle between $\vec{w}$ and the meridional plane is denoted φ_m . The relations

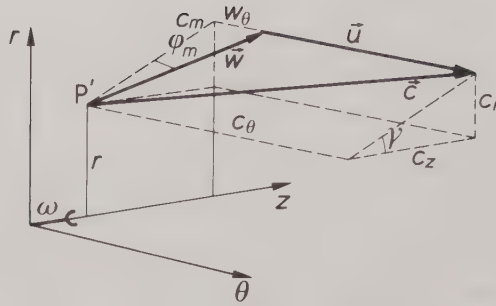


Fig. 3.3. The velocity $\vec{c}$ and its various components at the point P' in the meridional plane

between the velocities are

$$\left\{ \begin{array}{l} \vec{c} = \vec{w} + \vec{u} \\ \vec{w} = (w_r, w_\theta, w_z) \\ \vec{u} = \vec{\omega} \times \vec{r} = \omega r \hat{z} \times \hat{r} = \omega r \hat{\theta} = (0, \omega r, 0) \\ w_r = c_r \\ w_\theta = c_\theta - r\omega = c_m \tan \varphi_m \\ w_z = c_z \\ w^2 = w_\theta^2 + c_m^2 = c_m^2 / \cos^2 \varphi_m \end{array} \right. \quad (3.2)$$

The analysis may be facilitated if a coordinate system is used which matches the boundaries of the space under consideration. In Fig. 3.4 a curvilinear (i, k) system is introduced. The lines on which i is constant are mainly parallel to the through flow while the lines of constant k are straight and approximately perpendicular to the i-lines. These lines thus form a quasiorthogonal net, the boundary of which coincides with ACDBA. The lengths along the i- and k-lines from some points, O and S, see Fig. 3.4, are denoted v and s respectively. On a k-line r and z can be expressed

$$\left\{ \begin{array}{l} r = r_S + s \sin \alpha \\ z = z_S + s \cos \alpha \end{array} \right. \quad (3.3)$$

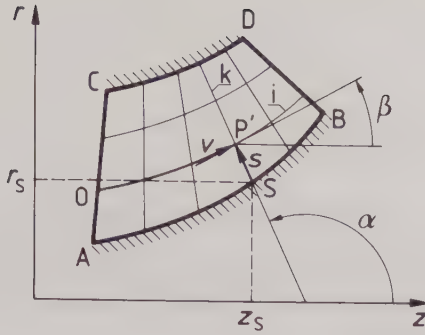


Fig. 3.4. The (i, k) coordinate system

where α is the angle between the z axis and the actual k -line.

The partial derivative of some function $N(r, z)$ with respect to a variable s is, according to the chain rule

$$\frac{\partial N}{\partial s} = \frac{\partial N}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial N}{\partial z} \frac{\partial z}{\partial s}$$

A displacement Δs along a k -line causes the displacements $\Delta r = \Delta s \sin \alpha$ and $\Delta z = \Delta s \cos \alpha$ in the radial and axial directions respectively, see Fig. 3.4. The derivative with respect to the variable v is obtained if α is replaced by β which denotes the angle between the z axis and the tangent to the i -line. The result is then

$$\begin{cases} \frac{\partial N}{\partial s} = \frac{\partial N}{\partial r} \sin \alpha + \frac{\partial N}{\partial z} \cos \alpha \\ \frac{\partial N}{\partial v} = \frac{\partial N}{\partial r} \sin \beta + \frac{\partial N}{\partial z} \cos \beta \end{cases} \quad (3.4)$$

or solved with respect to the r and z derivatives

$$\begin{cases} \frac{\partial N}{\partial r} = \frac{1}{\sin(\alpha - \beta)} \left(\frac{\partial N}{\partial s} \cos \beta - \frac{\partial N}{\partial v} \cos \alpha \right) \\ \frac{\partial N}{\partial z} = \frac{1}{\sin(\alpha - \beta)} \left(\frac{\partial N}{\partial v} \sin \alpha - \frac{\partial N}{\partial s} \sin \beta \right) \end{cases} \quad (3.5)$$

The meridional streamlines may be used as i-lines. In this particular case the angles β and γ are identical, see Fig. 3.2.b; furthermore the v could be exchanged for m which thus is the length of the meridional streamlines, that is

$$\begin{cases} v = m \\ \beta = \gamma \end{cases} \quad (3.6)$$

Note that the walls AB and CD are always meridional streamlines.

A vector $\hat{a}$ which is perpendicular to the k-lines and points in the through flow direction is defined in Fig. 3.5. The angle λ between $\hat{a}$ and c_m is

$$\lambda = \gamma - \left(\alpha - \frac{\pi}{2} \right) \quad (3.7)$$

and if $c_m > 0$ then $|\lambda| < \pi/2$.

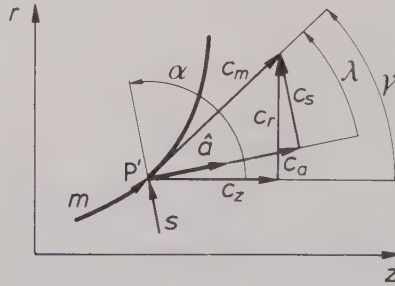


Fig. 3.5. The various velocities in the meridional plane

The meridional velocity c_m may be resolved in c_a and c_s which are the velocities in the $\hat{a}$ and s directions respectively. Fig. 3.5 and Eq. (3.7) yield

$$\begin{cases} c_a = c_r \sin(\gamma - \lambda) + c_z \cos(\gamma - \lambda) = -c_r \cos \alpha + c_z \sin \alpha = c_m \cos \lambda \\ c_s = c_r \cos(\gamma - \lambda) - c_z \sin(\gamma - \lambda) = c_r \sin \alpha + c_z \cos \alpha = c_m \sin \lambda \end{cases} \quad (3.8)$$

The angle φ_m , which is introduced in Fig. 3.3, between the meridional plane and the velocity $\vec{w}$, is due to the blades in the turbine. In Fig. 3.6 P and Q denote two adjacent points on the surface of a blade. Their projections on a meridional plane are P' and Q'. The angles PCP' and QDQ' then become Θ and $\Theta + \Delta\Theta$ respectively; furthermore a point Q'' is introduced so that the angle Q'DQ'' is Θ too. All the five points P, P', Q, Q', and Q'' lie on the surface of a cone with apex in T. The small triangle PQ''Q is right-angled and the angle QPQ'',

As the turbine under consideration is assumed to have an infinite number of blades, all fluid particles are forced to follow the blades exactly, that is the vectors $\vec{w}$ and $\vec{n}$ are perpendicular. With the aid of the Eqs. (3.1), (3.2), (3.9), (3.6), and (3.4) is obtained

$$\vec{n} \cdot \vec{w} = w_\theta - w_r \tan \varphi_r - w_z \tan \varphi_z = 0$$

$$w_\theta = c_m \sin \gamma r \frac{\partial \Theta}{\partial r} + c_m \cos \gamma r \frac{\partial \Theta}{\partial z} = c_m r \frac{\partial \Theta}{\partial m} = c_m \tan \varphi_m \quad (3.12)$$

The entire flow is thus known if the flow in the meridional plane is available. The problem, as previously stated, is reduced to a two-dimensional fictitious flow in the meridional plane. In a torque converter the meridional streamlines are closed curves which, when rotated about the z axis, generate toroids. The actual streamline is a type of screwline on the surface of such a toroid.

3.2. The Continuity Equation

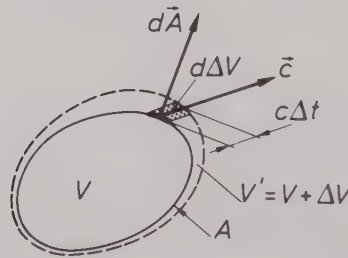


Fig. 3.7. Control volume

A concept of vital importance in the theory of fluid dynamics is "the control volume" which is a certain fixed domain in space. In Fig. 3.7 the control volume is denoted V and at the time t it contains a certain mass of fluid. At the later point of time $t + \Delta t$ this fluid has moved and occupies then the volume V' . Assume that x is some quantity known per unit mass. Then the total amount of x for all the fluid considered is: $X = \int x \, dm = \iiint x \, \rho \, dV$. The density of the fluid is denoted ρ . The change of X during the time Δt is, if x and ρ have simultaneously changed to $x + \Delta x$ and $\rho + \Delta \rho$ respectively

$$\Delta X = X_{t+\Delta t} - X_t = \iiint_{V'} (x + \Delta x) (\rho + \Delta \rho) dV - \iiint_V x \rho dV$$

The volume $\Delta V = V' - V$ is a thin coat on the volume V and, as shown in Fig. 3.7, its differential is: $d\Delta V = d\vec{A} \cdot (\vec{c} \Delta t)$ where A is the bounding surface of V and $\vec{A}$ a corresponding vector pointing outwards and perpendicular to the surface, while $\vec{c}$ is the velocity of the fluid at this boundary. Disregarding products of small quantities yields

$$\Delta X = \iiint_V (\rho \Delta x + x \Delta \rho) dV + \iiint_{\Delta V} x \rho dV = \iiint_V \Delta(\rho x) dV + \iint_A x \rho \Delta t \vec{c} \cdot d\vec{A}$$

which, when Δt approaches zero, gives the well-known expression for the material derivative

$$\frac{DX}{Dt} = \iiint_V \frac{\partial(x\rho)}{\partial t} dV + \iint_A x (\rho \vec{c} \cdot d\vec{A}) \quad (3.13)$$

If $x \equiv 1$, X would be the total mass of the certain mass of fluid which must be constant. This inserted in Eq. (3.13) gives

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \iint_A \rho \vec{c} \cdot d\vec{A} = 0 \quad (3.14)$$

which is the continuity equation in integral form. Applying the Gauss' theorem yields

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{c}) \right] dV = 0$$

and as the volume V can be chosen arbitrarily the integrand must be zero

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{c}) = \frac{\partial \rho}{\partial t} + \vec{c} \cdot \nabla \rho + \rho \nabla \cdot \vec{c} = 0 \quad (3.15)$$

which is the continuity equation in differential form. Using this and the Gauss' theorem Eq. (3.13) can be converted to

$$\frac{DX}{Dt} = \iiint_V \left[\frac{\partial x}{\partial t} + \vec{c} \cdot \nabla x \right] \rho dV \quad (3.16)$$

If the density of the fluid is assumed to be constant Eq. (3.15) is reduced to $\nabla \cdot \vec{c} = 0$ which in cylinder coordinates is

$$\frac{\partial(r c_r)}{\partial r} + \frac{\partial(r c_z)}{\partial z} = 0 \quad (3.17)$$

This equation is always satisfied if the velocities c_r and c_z are expressed as

$$\begin{cases} c_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z} \\ c_z = \frac{1}{r} \frac{\partial \Psi}{\partial r} \end{cases} \quad (3.18)$$

where Ψ is the Stokesian stream function. Its derivatives with respect to s and v are, with the aid of Eqs. (3.1) and (3.4)

$$\begin{cases} \frac{\partial \Psi}{\partial s} = r c_m \sin(\alpha - \gamma) \\ \frac{\partial \Psi}{\partial v} = r c_m \sin(\beta - \gamma) \end{cases}$$

Along a streamline Eq. (3.6) applies and hence $\partial \Psi / \partial m = 0$, that is, Ψ is constant on a streamline. Application of Eq. (3.7) gives

$$c_m = \frac{1}{r \cos \lambda} \frac{\partial \Psi}{\partial s} \quad (3.19)$$

which inserted in Eq. (3.8) gives the velocity in the $\hat{a}$ direction

$$c_a = \frac{1}{r} \frac{\partial \Psi}{\partial s} \quad (3.20)$$

The total flow of mass per unit time through the turbine is denoted $\dot{m}$. This quantity may be evaluated at any k -line giving

$$\dot{m} = \iint_{A_k} \rho \vec{c} \cdot d\vec{A} = \rho \iint_{A_k} c_a dA = 2\pi\rho \int_k c_a r ds$$

since the surface A_k is conical, see Fig. 3.4. The integration can be carried out if Eq. (3.20) is used

$$\dot{m} = 2\pi\rho (\psi_{CD} - \psi_{AB}) \quad (3.21)$$

The total through flow may also be expressed as volume per unit time. This quantity, denoted $\dot{V}$, is

$$\dot{V} = \frac{\dot{m}}{\rho} = 2\pi (\psi_{CD} - \psi_{AB}) \quad (3.22)$$

Eq. (3.13) can also be written

$$\frac{DX}{Dt} = \frac{\partial X_V}{\partial t} + \iint_A \mathbf{x} \, d\dot{m} \quad (3.23)$$

which is an integral form of Eq. (3.16).

Insertion of Eq. (3.1) in the continuity equation (3.17) gives

$$c_m \sin \gamma + r \frac{\partial (c_m \sin \gamma)}{\partial r} + r \frac{\partial (c_m \cos \gamma)}{\partial z} = 0$$

Evaluation of the derivatives and use of Eqs. (3.4), (3.5), and (3.6) yields

$$\frac{c_m \sin \gamma}{r} + \frac{\partial c_m}{\partial m} + \frac{c_m}{\sin (\alpha - \gamma)} \left(\frac{\partial \gamma}{\partial s} - \frac{\partial \gamma}{\partial m} \cos (\alpha - \gamma) \right) = 0$$

The curvature K and the radius of curvature R_K for the streamlines are defined

$$K = \frac{1}{R_K} = \frac{\partial \gamma}{\partial m} \quad (3.24)$$

which, together with Eq. (3.7), gives an expression for $\partial c_m / \partial m$

$$\frac{\partial c_m}{\partial m} = -\frac{c_m}{r} \sin \left(\lambda + \alpha - \frac{\pi}{2} \right) - \frac{c_m}{\cos \lambda} \frac{\partial \lambda}{\partial s} + K c_m \tan \lambda \quad (3.25)$$

3.3. Equations of Motion

Assume that the mass of fluid which is contained in a certain volume V at time t is exposed to a pressure p acting on the surface A of the volume and a force $\vec{f}$ per unit mass. A fluid particle dm has the momentum $\vec{c} \, dm$ and hence the total momentum for the fluid considered is: $\vec{P} = \iiint_V \vec{c} \, \rho \, dV$. Newton's second law of motion applied to this mass of fluid gives

$$\frac{D\vec{P}}{Dt} = \iiint_V \vec{f} \, \rho \, dV + \iint_A (-p) d\vec{A} \quad (3.26)$$

The last term can be converted to a volume integral with the aid of Gauss' theorem. The x component of this equation is, using Eq. (3.16)

$$\frac{DP_x}{Dt} = \iiint_V \left[\frac{\partial c_x}{\partial t} + \vec{c} \cdot \nabla c_x \right] \rho \, dV = \iiint_V \left(f_x \rho - \frac{\partial p}{\partial x} \right) dV$$

The y and z components are analogous to this; furthermore the volume V may be chosen arbitrarily which implies that the integrands are equal, thus yielding the well-known Euler's equation

$$\frac{\partial \vec{c}}{\partial t} + (\vec{c} \cdot \nabla) \vec{c} = - \frac{\nabla p}{\rho} + \vec{f} \quad (3.27)$$

The derivations of the equations of motion and continuity are dealt with in many textbooks, for instance Refs. [16] and [18], but are included here for the sake of completeness.

According to the rules of vector calculus, Eq. (3.27) may be written

$$\frac{\partial \vec{c}}{\partial t} + \nabla \left(\frac{c^2}{2} \right) - \vec{c} \times (\nabla \times \vec{c}) = - \frac{\nabla p}{\rho} + \vec{f} \quad (3.28)$$

The aim is to study stationary and frictionless flow but these restrictions are introduced later.

The vorticity $\vec{\xi}$ is defined as

$$\vec{\xi} = \nabla \times \vec{c} = \text{curl } \vec{c} \quad (3.29)$$

or, if the derivations are carried out and if the θ coordinate has no influence, as decided in Sec. 3.1

$$\left\{ \begin{array}{l} \xi_r = - \frac{1}{r} \frac{\partial (r c_\theta)}{\partial z} \\ \xi_\theta = \frac{\partial c_r}{\partial z} - \frac{\partial c_z}{\partial r} \\ \xi_z = \frac{1}{r} \frac{\partial (r c_\theta)}{\partial r} \end{array} \right. \quad (3.30)$$

Introduce furthermore the specific energy I as the sum of the pressure energy and the kinetic energy per unit mass

$$I = \frac{p}{\rho} + \frac{c^2}{2} \quad (3.31)$$

The Eq. (3.28) can now be written

$$\frac{\partial \vec{c}}{\partial t} + \vec{\xi} \times \vec{c} = - \nabla I + \vec{f} \quad (3.32)$$

or expanded

$$\begin{cases} \frac{\partial c_r}{\partial t} + \left(\frac{\partial c_r}{\partial z} - \frac{\partial c_z}{\partial r} \right) c_z - \frac{c_\theta}{r} \frac{\partial (r c_\theta)}{\partial r} = f_r - \frac{\partial I}{\partial r} \\ \frac{\partial c_\theta}{\partial t} + \frac{c_z}{r} \frac{\partial (r c_\theta)}{\partial z} + \frac{c_r}{r} \frac{\partial (r c_\theta)}{\partial r} = f_\theta \\ \frac{\partial c_z}{\partial t} - \frac{c_\theta}{r} \frac{\partial (r c_\theta)}{\partial z} - \left(\frac{\partial c_r}{\partial z} - \frac{\partial c_z}{\partial r} \right) c_r = f_z - \frac{\partial I}{\partial z} \end{cases} \quad (3.33)$$

Define further the specific moment of momentum as $\Gamma = r c_\theta$. Some expressions for this are obtained by means of Eqs. (3.1), (3.2), (3.12), (3.9), (3.6), (3.4), and (3.18).

$$\begin{cases} \Gamma = r c_\theta = r^2 \omega + r w_\theta = r^2 \omega + r c_m \tan \varphi_m \\ \Gamma = r^2 \omega + r^2 c_m \frac{\partial \Theta}{\partial m} = r^2 \omega + r^2 \left(c_r \frac{\partial \Theta}{\partial r} + c_z \frac{\partial \Theta}{\partial z} \right) \\ \Gamma = r^2 \omega + r \frac{\partial \Theta}{\partial z} \frac{\partial \Psi}{\partial r} - r \frac{\partial \Theta}{\partial r} \frac{\partial \Psi}{\partial z} \end{cases} \quad (3.34)$$

The θ component of Eq. (3.33) may now be rewritten

$$\frac{\partial \Gamma}{\partial t} + c_r \frac{\partial \Gamma}{\partial r} + c_z \frac{\partial \Gamma}{\partial z} = \frac{\partial \Gamma}{\partial t} + c_m \frac{\partial \Gamma}{\partial m} = r f_\theta \quad (3.35)$$

If the flow is stationary and if no forces exist then Γ is constant along the streamlines.

The field of forces $\vec{f}$ is split into two parts $\vec{k}$ and $\vec{d}$, which are perpendicular and parallel to the blades respectively

$$\vec{f} = \vec{k} + \vec{d} \quad (3.36)$$

As the vector $\vec{n}$, Eq. (3.11), is perpendicular to the blades the force $\vec{k}$ may be written

$$\begin{cases} \vec{k} = k_\theta \vec{n} = (-k_\theta \tan \varphi_r, k_\theta, -k_\theta \tan \varphi_z) \\ k_r = -k_\theta \tan \varphi_r \\ k_z = -k_\theta \tan \varphi_z \end{cases} \quad (3.37)$$

Forming the dot product between Eq. (3.32) and the velocity $\vec{c}$ yields

$$\frac{\partial}{\partial t} \left(\frac{c^2}{2} \right) = -\vec{c} \cdot \nabla I + \vec{c} \cdot \vec{f} \quad (3.38)$$

which, using Eqs. (3.1), (3.2), (3.6), (3.4), (3.36), (3.37), (3.12), and (3.35), may be written

$$\frac{\partial}{\partial t} \left(\frac{c^2}{2} \right) + c_m \frac{\partial I}{\partial m} - \vec{w} \cdot \vec{d} - \omega \left(\frac{\partial \Gamma}{\partial t} + c_m \frac{\partial \Gamma}{\partial m} \right) = 0$$

Noting that $c^2 = c_m^2 + c_\theta^2 = c_m^2 + \Gamma^2/r^2$ and introducing the rothalpy $H = I - \omega\Gamma$ gives

$$c_m \frac{\partial c_m}{\partial t} + \left(\frac{\Gamma}{r^2} - \omega \right) \frac{\partial \Gamma}{\partial t} + c_m \frac{\partial H}{\partial m} = \vec{w} \cdot \vec{d}$$

This equation shows that the rothalpy is constant along the streamlines for stationary and non-dissipative flow. The rothalpy may be written with the aid of Eqs. (3.31), (3.34), and (3.2)

$$\begin{cases} H = I - \omega\Gamma \\ H = \frac{p}{\rho} + \frac{c^2}{2} - \omega\Gamma = \frac{p}{\rho} + \frac{w^2}{2} - \frac{\omega^2 r^2}{2} \end{cases} \quad (3.39)$$

and its derivative with respect to the coordinate m is

$$\frac{\partial H}{\partial m} = \frac{\vec{w} \cdot \vec{d}}{c_m} - r \tan \varphi_m \frac{\partial \omega}{\partial t} - (1 + \tan^2 \varphi_m) \frac{\partial c_m}{\partial t} - c_m \tan \varphi_m \frac{\partial \tan \varphi_m}{\partial t} \quad (3.40)$$

if Γ is eliminated by Eq. (3.34).

Assume that the force $\vec{d}$ opposes the flow relative to the turbine $\vec{w}$, that is

$$\vec{d} = -\Phi \vec{w}, \quad \Phi > 0 \quad (3.41)$$

For stationary flow Eq. (3.40) then becomes

$$\frac{\partial H}{\partial m} = -\frac{\Phi w^2}{c_m} = -\Phi c_m (1 + \tan^2 \varphi_m) \quad (3.42)$$

The rothalpy is declining along the streamlines for dissipative flow. Further manipulation of the equations of motion (3.33) for the r and z directions yields

$$\begin{cases} \frac{\partial c_r}{\partial t} + \zeta_\theta c_z - \frac{w_\theta}{r} \frac{\partial \Gamma}{\partial r} = - \frac{\partial H}{\partial r} + f_r \\ \frac{\partial c_z}{\partial t} - \zeta_\theta c_r - \frac{w_\theta}{r} \frac{\partial \Gamma}{\partial z} = - \frac{\partial H}{\partial z} + f_z \end{cases} \quad (3.43)$$

and using Eqs. (3.36), (3.37), (3.9), and (3.41) for the forces

$$\begin{cases} \frac{\partial c_r}{\partial t} + \zeta_\theta c_z - \frac{w_\theta}{r} \frac{\partial \Gamma}{\partial r} = - \frac{\partial H}{\partial r} - k_\theta r \frac{\partial \Theta}{\partial r} - \Phi c_r \\ \frac{\partial c_z}{\partial t} - \zeta_\theta c_r - \frac{w_\theta}{r} \frac{\partial \Gamma}{\partial z} = - \frac{\partial H}{\partial z} - k_\theta r \frac{\partial \Theta}{\partial z} - \Phi c_z \end{cases} \quad (3.44)$$

In the absence of blades the force $\vec{k}$ does not exist, so that the fluid in this case is only affected by the dissipative force. Eq. (3.35) becomes now, with the aid of Eqs. (3.41) and (3.34)

$$\frac{\partial \Gamma}{\partial t} + c_m \frac{\partial \Gamma}{\partial m} = r d_\theta = -r \Phi w_\theta = -\Phi (\Gamma - r^2 \omega) \quad (3.45)$$

Furthermore, the angle φ_m may now be defined as

$$\tan \varphi_m = \frac{w_\theta}{c_m} = \frac{\Gamma - r^2 \omega}{r c_m} \quad (3.46)$$

while the other blade angles do not exist.

3.4. Some Expressions for the Vorticity ζ_θ

The θ component of the vorticity ζ_θ may be expressed in two ways, namely by the equations of motion (3.44) and by the definition (3.30). These two variants are denoted $\zeta_{\theta,F}$ and $\zeta_{\theta,\Psi}$ respectively and are of course equal, that is

$$\zeta_\theta = \zeta_{\theta,F} = \zeta_{\theta,\Psi} \quad (3.47)$$

Multiplying the Eqs. (3.44) by c_z and $-c_r$ respectively and then adding them

yields

$$\begin{aligned} & \left(c_z \frac{\partial c_r}{\partial t} - c_r \frac{\partial c_z}{\partial t} \right) + \zeta_{\theta,F} (c_z^2 + c_r^2) - \frac{w_\theta}{r} \left(c_z \frac{\partial \Gamma}{\partial r} - c_r \frac{\partial \Gamma}{\partial z} \right) = \\ & = - \left(c_z \frac{\partial H}{\partial r} - c_r \frac{\partial H}{\partial z} \right) - k_\theta r \left(c_z \frac{\partial \Theta}{\partial r} - c_r \frac{\partial \Theta}{\partial z} \right) \end{aligned} \quad (3.48)$$

Elimination of w_θ and k_θ using Eqs. (3.34), (3.36), (3.35), and (3.41) and if stationary flow is considered

$$\begin{aligned} c_m^2 \zeta_{\theta,F} = & \left(c_r \frac{\partial \Theta}{\partial r} + c_z \frac{\partial \Theta}{\partial z} \right) \left(c_z \frac{\partial \Gamma}{\partial r} - c_r \frac{\partial \Gamma}{\partial z} \right) - \left(c_z \frac{\partial H}{\partial r} - c_r \frac{\partial H}{\partial z} \right) \\ & - \left(c_r \frac{\partial \Gamma}{\partial r} + c_z \frac{\partial \Gamma}{\partial z} + r \Phi w_\theta \right) \left(c_z \frac{\partial \Theta}{\partial r} - c_r \frac{\partial \Theta}{\partial z} \right) \end{aligned}$$

and further manipulations using Eqs. (3.2) and (3.18) give

$$\begin{aligned} \zeta_{\theta,F} = & \left(\frac{\partial \Theta}{\partial z} \frac{\partial \Gamma}{\partial r} - \frac{\partial \Theta}{\partial r} \frac{\partial \Gamma}{\partial z} \right) - \frac{1}{r c_m^2} \left(\frac{\partial \Psi}{\partial r} \frac{\partial H}{\partial r} + \frac{\partial \Psi}{\partial z} \frac{\partial H}{\partial z} \right) \\ & - \frac{\Phi \tan \varphi_m}{c_m} \left(\frac{\partial \Psi}{\partial r} \frac{\partial \Theta}{\partial r} + \frac{\partial \Psi}{\partial z} \frac{\partial \Theta}{\partial z} \right) \end{aligned}$$

Define

$$\begin{cases} \chi[A, B] = \frac{\partial A}{\partial r} \frac{\partial B}{\partial z} - \frac{\partial A}{\partial z} \frac{\partial B}{\partial r} \\ \Upsilon[A, B] = \frac{\partial A}{\partial r} \frac{\partial B}{\partial r} + \frac{\partial A}{\partial z} \frac{\partial B}{\partial z} \end{cases} \quad (3.49)$$

and the vorticity may be written as

$$\zeta_{\theta,F} = \chi[\Gamma, \Theta] - \frac{1}{r c_m^2} \Upsilon[\Psi, H] - \frac{\Phi \tan \varphi_m}{c_m} \Upsilon[\Psi, \Theta] \quad (3.50)$$

and the specific moment of momentum Γ , Eq. (3.34), as

$$\Gamma = r^2 \omega + r \chi[\Psi, \Theta] \quad (3.51)$$

χ and Υ and hence $\zeta_{\theta,F}$ and Γ may be expressed in different coordinates. Substitution by Eq. (3.5) gives

$$\left\{ \begin{aligned} \chi[A, B] &= \frac{1}{\sin(\alpha - \beta)} \left(\frac{\partial A}{\partial s} \frac{\partial B}{\partial v} - \frac{\partial A}{\partial v} \frac{\partial B}{\partial s} \right) \\ \Upsilon[A, B] &= \frac{1}{\sin^2(\alpha - \beta)} \left(\frac{\partial A}{\partial s} \frac{\partial B}{\partial s} + \frac{\partial A}{\partial v} \frac{\partial B}{\partial v} \right) \\ &\quad - \frac{\cos(\alpha - \beta)}{\sin^2(\alpha - \beta)} \left(\frac{\partial A}{\partial v} \frac{\partial B}{\partial s} + \frac{\partial A}{\partial s} \frac{\partial B}{\partial v} \right) \end{aligned} \right. \quad (3.52)$$

A special case of interest is $A = \Psi$ and $m = v$ which, together with Eqs. (3.6), (3.7), and (3.19), yields

$$\left\{ \begin{aligned} \chi[\Psi, B] &= r c_m \frac{\partial B}{\partial m} \\ \Upsilon[\Psi, B] &= \frac{r c_m}{\cos \lambda} \frac{\partial B}{\partial s} - r c_m \tan \lambda \frac{\partial B}{\partial m} \end{aligned} \right. \quad (3.53)$$

Now Eq. (3.50) may be rewritten with the aid of Eqs. (3.42) and (3.9)

$$\zeta_{\theta, F} = \chi[\Gamma, \Theta] - \frac{1}{c_m \cos \lambda} \frac{\partial H}{\partial s} - \Phi \frac{\sin \lambda + \tan \varphi_s \tan \varphi_m}{\cos \lambda} \quad (3.54)$$

or

$$\zeta_{\theta, F} = \frac{1}{r \cos \lambda} \left(\frac{\partial \Gamma}{\partial s} \tan \varphi_m - \frac{\partial \Gamma}{\partial m} \tan \varphi_s \right) - \frac{1}{c_m \cos \lambda} \frac{\partial H}{\partial s} - \Phi \frac{\sin \lambda + \tan \varphi_s \tan \varphi_m}{\cos \lambda} \quad (3.55)$$

If no blades are present this is imposed by dropping the force k_θ , changing Eq. (3.48) to

$$\zeta_{\theta, F} = \frac{w_\theta}{r^2 c_m^2} \Upsilon[\Psi, \Gamma] - \frac{1}{r c_m^2} \Upsilon[\Psi, H] \quad (3.56)$$

and the use of Eqs. (3.53), (3.45), (3.46), and (3.42) yields

$$\zeta_{\theta, F} = \frac{\Gamma - r^2 \omega}{r^2 c_m \cos \lambda} \frac{\partial \Gamma}{\partial s} - \frac{1}{c_m \cos \lambda} \frac{\partial H}{\partial s} - \Phi \tan \lambda \quad (3.57)$$

A comparison between Eqs. (3.55) and (3.57) yields the fact that the latter may

be obtained from the former by inserting $\varphi_s = 0$ and the φ_m defined in Eq. (3.46).

As previously stated, the rothalpy is constant along a certain streamline, that is the rothalpy is primarily a function of Ψ only, $H = H(\Psi(r, z))$, for non-dissipative flow. Its derivative with respect to the coordinate s is, as $\partial r/\partial s = \sin \alpha$ and $\partial z/\partial s = \cos \alpha$

$$\frac{\partial H}{\partial s} = \frac{dH}{d\Psi} \frac{\partial \Psi}{\partial r} \sin \alpha + \frac{dH}{d\Psi} \frac{\partial \Psi}{\partial z} \cos \alpha = r c_m \cos \lambda \frac{dH}{d\Psi} \quad (3.58)$$

where Eqs. (3.18) and (3.8) have also been used. This derivation is applicable to Γ too when this is constant along the streamlines, which is the case when neither blades nor friction exist. Eqs. (3.54) and (3.57) may now be written

$$\zeta_{\theta,F} = \chi[\Gamma, \Theta] - r \frac{dH}{d\Psi} \quad (3.59)$$

$$\zeta_{\theta,F} = \frac{\Gamma - r^2 \omega}{r} \frac{d\Gamma}{d\Psi} - r \frac{dH}{d\Psi} = \frac{\Gamma}{r} \frac{d\Gamma}{d\Psi} - r \frac{dI}{d\Psi} \quad (3.60)$$

and if Eq. (3.51) is inserted in (3.59)

$$\zeta_{\theta,F} = \chi[r^2 \omega + r \chi[\Psi, \Theta], \Theta] - r \frac{dH}{d\Psi} \quad (3.61)$$

Carrying out the derivations gives

$$\begin{aligned} \zeta_{\theta,F} = & \frac{\partial^2 \Psi}{\partial r^2} r \left(\frac{\partial \Theta}{\partial z} \right)^2 - 2 \frac{\partial^2 \Psi}{\partial r \partial z} r \frac{\partial \Theta}{\partial r} \frac{\partial \Theta}{\partial z} + \frac{\partial^2 \Psi}{\partial z^2} r \left(\frac{\partial \Theta}{\partial r} \right)^2 \\ & + \frac{\partial \Psi}{\partial r} \left(r \frac{\partial^2 \Theta}{\partial r \partial z} \frac{\partial \Theta}{\partial z} + \left(\frac{\partial \Theta}{\partial z} \right)^2 - r \frac{\partial^2 \Theta}{\partial z^2} \frac{\partial \Theta}{\partial r} \right) \\ & + \frac{\partial \Psi}{\partial z} \left(r \frac{\partial^2 \Theta}{\partial r \partial z} \frac{\partial \Theta}{\partial r} - \frac{\partial \Theta}{\partial r} \frac{\partial \Theta}{\partial z} - r \frac{\partial^2 \Theta}{\partial r^2} \frac{\partial \Theta}{\partial z} \right) \\ & + 2r\omega \frac{\partial \Theta}{\partial z} - r \frac{dH}{d\Psi} \end{aligned} \quad (3.62)$$

An alternative expression is obtained if Eqs. (3.55), (3.34), and (3.25) are combined

$$\begin{aligned}
\zeta_{\theta, F} = & \frac{1}{\cos \lambda} \left\{ \frac{\partial c_m}{\partial s} \tan^2 \varphi_m + c_m \frac{\tan \varphi_s \tan \varphi_m}{\cos \lambda} \frac{\partial \lambda}{\partial s} \right. \\
& - c_m K \tan \varphi_s \tan \varphi_m \tan \lambda + c_m \frac{\sin \alpha \tan^2 \varphi_m}{r} \\
& + c_m \left(\tan \varphi_m \frac{\partial \tan \varphi_m}{\partial s} - \tan \varphi_s \frac{\partial \tan \varphi_m}{\partial m} \right) \\
& + 2 \omega (\sin \alpha \tan \varphi_m - \sin \gamma \tan \varphi_s) \\
& \left. - \frac{1}{c_m} \frac{\partial H}{\partial s} - \Phi (\sin \lambda + \tan \varphi_s \tan \varphi_m) \right\} \quad (3.63)
\end{aligned}$$

where also the derivative $\partial r / \partial m = \sin \gamma$ has been used.

The other branch of the vorticity, $\zeta_{\theta, \Psi}$, is obtained by inserting Eq. (3.18) in (3.30)

$$\zeta_{\theta, \Psi} = - \left(\frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \Psi}{\partial z} \right) + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \right) \quad (3.64)$$

or alternatively with the Eqs. (3.18), (3.1), (3.5), (3.6), (3.7), (3.24), and (3.25)

$$\zeta_{\theta, \Psi} = \frac{1}{\cos \lambda} \left\{ - \frac{\partial c_m}{\partial s} + c_m \left(\frac{K}{\cos \lambda} - \tan \lambda \frac{\partial \lambda}{\partial s} - \frac{\sin \gamma \sin \lambda}{r} \right) \right\} \quad (3.65)$$

Insertion of Eqs. (3.63), (3.65), and (3.42) in (3.47) yields

$$\begin{aligned}
\frac{\partial c_m}{\partial s} + A c_m + B + \frac{C}{c_m} &= 0 \\
A = & \frac{1}{1 + \tan^2 \varphi_m} \left\{ K \left(- \frac{1}{\cos \lambda} - \tan \lambda \tan \varphi_s \tan \varphi_m \right) \right. \\
& + \frac{\partial \lambda}{\partial s} \left(\tan \lambda + \frac{\tan \varphi_s \tan \varphi_m}{\cos \lambda} \right) + \tan \varphi_m \frac{\partial \tan \varphi_m}{\partial s} - \tan \varphi_s \frac{\partial \tan \varphi_m}{\partial m} \\
& \left. + \frac{\sin \lambda \sin \gamma + \sin \alpha \tan^2 \varphi_m}{r} \right\}
\end{aligned}$$

$$B = \frac{2\omega}{1 + \tan^2 \varphi_m} (\sin \alpha \tan \varphi_m - \sin \gamma \tan \varphi_s)$$

$$C = \frac{1}{1 + \tan^2 \varphi_m} \left(-\frac{\partial H}{\partial s} + \frac{\partial H}{\partial m} \frac{\sin \lambda + \tan \varphi_s \tan \varphi_m}{1 + \tan^2 \varphi_m} \right) \quad (3.66)$$

or if the Eqs. (3.62) and (3.64) are inserted in (3.47) instead

$$\begin{aligned} & \frac{\partial^2 \Psi}{\partial r^2} \left(\frac{1}{r} + r \left(\frac{\partial \Theta}{\partial z} \right)^2 \right) - 2 \frac{\partial^2 \Psi}{\partial r \partial z} r \frac{\partial \Theta}{\partial r} \frac{\partial \Theta}{\partial z} + \frac{\partial^2 \Psi}{\partial z^2} \left(\frac{1}{r} + r \left(\frac{\partial \Theta}{\partial r} \right)^2 \right) \\ & + \frac{\partial \Psi}{\partial r} \left(-\frac{1}{r^2} + r \frac{\partial^2 \Theta}{\partial r \partial z} \frac{\partial \Theta}{\partial z} + \left(\frac{\partial \Theta}{\partial z} \right)^2 - r \frac{\partial^2 \Theta}{\partial z^2} \frac{\partial \Theta}{\partial r} \right) \\ & + \frac{\partial \Psi}{\partial z} \left(r \frac{\partial^2 \Theta}{\partial r \partial z} \frac{\partial \Theta}{\partial r} - \frac{\partial \Theta}{\partial r} \frac{\partial \Theta}{\partial z} - r \frac{\partial^2 \Theta}{\partial r^2} \frac{\partial \Theta}{\partial z} \right) + 2r\omega \frac{\partial \Theta}{\partial z} - r \frac{dH}{d\Psi} = 0 \end{aligned} \quad (3.67)$$

which is valid for non-dissipative flow. This is a linear second order partial differential equation for Ψ if the function $dH/d\Psi$ is linear with respect to Ψ . A differential equation of the type

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u + g = 0$$

where the subscripts allude to the derivatives of the function u , is classified according to the sign of the quantity $ac - b^2$. The differential equation is said to be hyperbolic, parabolic, or elliptic if $ac - b^2$ is negative, zero, or positive respectively. For Eq. (3.67) holds

$$ac - b^2 = \frac{1}{r^2} + \left(\frac{\partial \Theta}{\partial r} \right)^2 + \left(\frac{\partial \Theta}{\partial z} \right)^2 = \frac{1 + \tan^2 \varphi_r + \tan^2 \varphi_z}{r^2} > 0 \quad (3.68)$$

and hence it is always elliptic.

The specific energy and hence the rothalpy is eliminated from Eq. (3.32) by applying the operator $\nabla \times$, giving

$$\nabla \times \frac{\partial \vec{c}}{\partial t} + \nabla \times (\vec{\xi} \times \vec{c}) - \nabla \times \vec{f} = 0$$

and performing the derivations and using the Eqs. (3.30) and (3.34) yields

$$\left\{ \begin{aligned} & - \frac{\partial^2 \Gamma}{\partial z \partial t} - \frac{\partial}{\partial z} \left(r \left(\frac{c_z}{r} \frac{\partial \Gamma}{\partial z} + \frac{c_r}{r} \frac{\partial \Gamma}{\partial r} \right) \right) + \frac{\partial(r f_\theta)}{\partial z} = 0 \\ & \frac{\partial \xi_\theta}{\partial t} - \left\{ \frac{\partial}{\partial r} \left(- \frac{\Gamma}{r^2} \frac{\partial \Gamma}{\partial z} - \xi_\theta c_r \right) - \frac{\partial}{\partial z} \left(\xi_\theta c_z - \frac{\Gamma}{r^2} \frac{\partial \Gamma}{\partial r} \right) \right\} + \\ & + \frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z} = 0 \\ & \frac{\partial^2 \Gamma}{\partial r \partial t} + \frac{\partial}{\partial r} \left(r \left(\frac{c_z}{r} \frac{\partial \Gamma}{\partial z} + \frac{c_r}{r} \frac{\partial \Gamma}{\partial r} \right) \right) - \frac{\partial(r f_\theta)}{\partial r} = 0 \end{aligned} \right. \quad (3.69)$$

Comparison with Eq. (3.35) shows that the r and z components of (3.69) are derivatives of that equation, whereas the θ component is

$$\begin{aligned} & \frac{\partial \xi_\theta}{\partial t} + \frac{\partial}{\partial r} \left(\frac{1}{r^2} \right) \Gamma \frac{\partial \Gamma}{\partial z} + c_r \frac{\partial \xi_\theta}{\partial r} + c_z \frac{\partial \xi_\theta}{\partial z} + \xi_\theta \left(\frac{\partial c_r}{\partial r} + \frac{\partial c_z}{\partial z} \right) + \\ & + \frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z} = 0
 \end{aligned}$$

which may be rewritten with the aid of Eq. (3.17)

$$\frac{1}{r} \frac{\partial \xi_\theta}{\partial t} + c_r \frac{\partial}{\partial r} \left(\frac{\xi_\theta}{r} \right) + c_z \frac{\partial}{\partial z} \left(\frac{\xi_\theta}{r} \right) = \frac{2\Gamma}{r^4} \frac{\partial \Gamma}{\partial z} - \frac{1}{r} \left(\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z} \right)$$

Insertion of the Eqs. (3.36), (3.37), (3.9), and (3.41) gives, finally

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\xi_\theta}{r} \right) + c_m \frac{\partial}{\partial m} \left(\frac{\xi_\theta}{r} \right) &= \frac{2\Gamma}{r^4} \frac{\partial \Gamma}{\partial z} + \frac{1}{r} \left(\frac{\partial(k_\theta r)}{\partial r} \frac{\partial \Theta}{\partial z} - \frac{\partial(k_\theta r)}{\partial z} \frac{\partial \Theta}{\partial r} \right) \\ &+ \frac{1}{r} \left(c_z \frac{\partial \Phi}{\partial r} - c_r \frac{\partial \Phi}{\partial z} \right) + \frac{\Phi}{r} \left(\frac{\partial c_z}{\partial r} - \frac{\partial c_r}{\partial z} \right) \end{aligned} \quad (3.70)$$

or using Eqs. (3.18), (3.30), and (3.49)

$$\frac{\partial}{\partial t} \left(\frac{\xi_\theta}{r} \right) + c_m \frac{\partial}{\partial m} \left(\frac{\xi_\theta}{r} \right) = \frac{2\Gamma}{r^4} \frac{\partial \Gamma}{\partial z} + \frac{1}{r} \chi[k_\theta r, \Theta] + \frac{1}{r^2} \Upsilon[\Psi, \Phi] - \frac{\Phi}{r} \quad (3.71)$$

The force k_θ or alternatively the torque $k_\theta r$ may be expressed by combining Eqs. (3.35), (3.41), and (3.34)

$$k_\theta r = \frac{\partial \Gamma}{\partial t} + c_r \frac{\partial \Gamma}{\partial r} + c_z \frac{\partial \Gamma}{\partial z} + \Phi (\Gamma - r^2 \omega) \quad (3.72)$$

or

$$k_\theta r = \frac{\partial \Gamma}{\partial t} + \frac{1}{r} \chi[\Psi, \Gamma] + \Phi (\Gamma - r^2 \omega) \quad (3.73)$$

This and Eqs. (3.51) and (3.64) inserted in Eq. (3.71) gives for stationary and non-dissipative flow

$$\begin{aligned} \chi[\Psi, -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \Psi}{\partial z} \right)] = \\ 2 \left(\omega + \frac{1}{r} \chi[\Psi, \Theta] \right) \frac{\partial}{\partial z} \chi[\Psi, \Theta] + \chi \left[\frac{1}{r} \chi[\Psi, r^2 \omega + r \chi[\Psi, \Theta]], \Theta \right] \end{aligned} \quad (3.74)$$

which is a non-linear third order partial differential equation. This equation is dealt with in Ref. [12] to some extent.

Eq. (3.67) may also be considered as a differential equation for Θ if Ψ is known. This is a non-linear differential equation but if the classification is still done it yields

$$\begin{aligned} ac - b^2 &= \left(-\frac{\partial \Psi}{\partial z} r \frac{\partial \Theta}{\partial z} \right) \left(-\frac{\partial \Psi}{\partial r} r \frac{\partial \Theta}{\partial r} \right) - \\ &\quad - \left(\frac{1}{2} \left(\frac{\partial \Psi}{\partial r} r \frac{\partial \Theta}{\partial z} + \frac{\partial \Psi}{\partial z} r \frac{\partial \Theta}{\partial r} \right) \right)^2 \\ &= -\frac{r^2}{4} \left(\frac{\partial \Psi}{\partial r} \frac{\partial \Theta}{\partial z} - \frac{\partial \Psi}{\partial z} \frac{\partial \Theta}{\partial r} \right)^2 = -\frac{(\Gamma - r^2 \omega)^2}{4} \leq 0 \end{aligned} \quad (3.75)$$

and hence this differential equation is usually hyperbolic.

3.5. Torque, Power, and Axial Thrust

The torque in the shaft of the turbine is a quantity of major interest. A volume dV of the fluid is subjected to a force $\rho \vec{f} dV$ due to the blades. Its torque is thus $\vec{r} \times \rho \vec{f} dV$, the z component of which is directed along the shaft, that is: $dM_z = \hat{z} \cdot (\vec{r} \times \rho \vec{f}) dV = \rho (\hat{z} \times \vec{r}) \cdot \vec{f} dV = \rho r f_\theta dV$. The forces due to the pressure, $p d\vec{A}$, have no circumferential components, as the actual surfaces are annular, see Fig. 3.1, and hence the pressure does not contribute to the torque M_z

$$M_z = \rho \iiint_V f_\theta r dV \quad (3.76)$$

The equation of motion (3.26) may also be written

$$\frac{D\vec{L}}{Dt} = \iiint_V \vec{r} \times \vec{f} \rho dV + \iint_A \vec{r} \times (-p) d\vec{A} \quad (3.77)$$

where $\vec{L}$ is the moment of momentum: $\vec{L} = \iiint_V \vec{r} \times \vec{c} \rho dV$. This may be expressed with Eq. (3.23); furthermore only the z component of $\vec{L}$ is useful here. Neither of the vectors $\vec{r}$ and $d\vec{A}$ has any circumferential component, so their cross product only has a circumferential component. Eq. (3.77) may now be written, as $L_z = \hat{z} \cdot \vec{L}$ and $\hat{z} \cdot (\vec{r} \times \vec{c}) = (\hat{z} \times \vec{r}) \cdot \vec{c} = r c_\theta = \Gamma$

$$\rho \iiint_V \frac{\partial \Gamma}{\partial t} dV + \iint_A \Gamma d\dot{m} = \rho \iiint_V r f_\theta dV$$

which is an integral form of Eq. (3.35). As $d\dot{m} = \rho \vec{c} \cdot d\vec{A}$ this quantity is positive where the discharge occurs, that is at the BD surface, Fig. 3.1, and negative at the inlet AC. Differentiation of Eq. (3.21) gives $d\dot{m} = 2\pi \rho d\Psi$ and the torque is

$$M_z = \rho \iiint_V \frac{\partial \Gamma}{\partial t} dV + 2\pi \rho \int_{\psi_{AB}}^{\psi_{CD}} \Gamma_{BD} d\Psi - 2\pi \rho \int_{\psi_{AB}}^{\psi_{CD}} \Gamma_{AC} d\Psi \quad (3.78)$$

Introducing the axial moment of momentum as

$$L_z = 2\pi \rho \int_{\psi_{AB}}^{\psi_{CD}} \Gamma d\Psi \quad (3.79)$$

allows the torque for stationary flow to be written

$$M_z = L_{z,BD} - L_{z,AC} \quad (3.80)$$

By considering the z component of the equation of motion (3.26) the axial thrust on the turbine is found

$$\frac{DP_z}{Dt} = \iiint_V \frac{\partial c_z}{\partial t} \rho dV + \iint_A c_z d\dot{m} = F_z - \iint_A p \hat{z} \cdot d\vec{A} \quad (3.81)$$

where F_z is the axial force due to the blades

$$F_z = \rho \iiint_V f_z dV \quad (3.82)$$

According to Fig. 3.5 the dot product $\hat{z} \cdot d\vec{A}$ is $dA \cos(\alpha - \frac{\pi}{2}) = dA \sin \alpha$ on a k -line, and from Fig. 3.4 it appears that $dA = 2\pi r ds$; furthermore $ds \sin \alpha = dr$. The force due to the pressure on a k -line is thus

$$F_{12} = -2\pi \int_{s_1}^{s_2} p \sin \alpha r ds = -2\pi \int_{r_1}^{r_2} p r dr \quad (3.83)$$

The force is assumed to act on the fluid causing the minus sign. The axial momentum is defined analogous to the axial moment of momentum, Eq. (3.79)

$$J_z = 2\pi \rho \int_{\psi_{AB}}^{\psi_{CD}} c_z d\psi \quad (3.84)$$

and for stationary flow Eq. (3.81) is

$$J_{z,BD} - J_{z,AC} = F_z + F_{AB} + F_{BD} + F_{DC} + F_{CA} \quad (3.85)$$

Eq. (3.83) is also applicable to the i -lines, AB and DC; here s and α are changed to m and γ respectively. Notice the importance of the order of the subscripts in Eqs. (3.83) and (3.85). The forces acting on the turbine itself are $-F_z$ due to the blades and $-F_{AB}$ and $-F_{DC}$ due to the walls AB and CD and hence the axial force on the turbine F_{ax} is

$$F_{ax} = F_z + F_{AB} + F_{DC}$$

or using Eq. (3.85)

$$F_{ax} = (J_{z,BD} - J_{z,AC}) - (F_{BD} + F_{CA}) \quad (3.86)$$

which is the axial thrust on the shaft of the turbine caused by the flow in the area ABCD. Generally additional forces must be considered due to pressures on other

parts of the turbine.

Finally the power of the field of forces is: $\iiint \vec{f} \cdot \vec{c} \rho dV$ which may be evaluated either by Eq. (3.38) or by Eqs. (3.36), (3.12), and (3.2).

$$\rho \iiint_V \vec{f} \cdot \vec{c} dV = \rho \iiint_V \left(\frac{\partial}{\partial t} \left(\frac{c^2}{2} \right) + \vec{c} \cdot \nabla I \right) dV = \rho \iiint_V (\vec{d} \cdot \vec{w} + \vec{f} \cdot \omega r \hat{\theta}) dV$$

Identify the dissipative power E_D and the power on the shaft E

$$E_D = -\rho \iiint_V \vec{d} \cdot \vec{w} dV = \rho \iiint_V \Phi w^2 dV \quad (3.87)$$

$$E = \rho \omega \iiint_V r f_\theta dV = M_z \omega \quad (3.88)$$

Furthermore, $\vec{c} \cdot \nabla I = \nabla \cdot (\vec{c} I)$ while $\nabla \cdot \vec{c} = 0$, Eq. (3.17), and hence

$$E - E_D = \rho \iiint_V \frac{\partial}{\partial t} \left(\frac{c^2}{2} \right) dV + \rho \iint_A I \vec{c} \cdot d\vec{A} \quad (3.89)$$

Analogously to the axial momentum and moment of momentum, Eqs. (3.84) and (3.79), a power flux may be introduced as

$$W = 2\pi\rho \int_{\psi_{AB}}^{\psi_{CD}} I d\psi \quad (3.90)$$

The non-stationary term in Eq. (3.89) is the time derivative of the total kinetic energy of the fluid in the volume V . For stationary flow holds

$$E - E_D = W_{BD} - W_{AC} \quad (3.91)$$

3.6. Non-Dimensional Quantities

It is convenient to use non-dimensional quantities when carrying out the numerical calculations. All coordinates are made non-dimensional through division by a significative measure of the torque converter, for instance its outer radius R_D , that is

$$r_0 = \frac{r}{R_D}; z_0 = \frac{z}{R_D}; s_0 = \frac{s}{R_D}; v_0 = \frac{v}{R_D}; m_0 = \frac{m}{R_D} \quad (3.92)$$

In a similar way the angular velocities are made non-dimensional with the aid of a typical value ω_p

$$\omega_0 = \frac{\omega}{\omega_p} \quad (3.93)$$

The non-dimensional velocities are now, see Eq. (3.2)

$$c_0 = \frac{c}{R_D \omega_p} ; w_0 = \frac{w}{R_D \omega_p} \quad (3.94)$$

Other non-dimensional quantities may be deduced in an analogous way by inspection of the equations. A list of them is given below.

Specific moment of momentum

$$\Gamma_0 = \frac{\Gamma}{R_D^2 \omega_p} \quad (3.95)$$

Stream function

$$\Psi_0 = \frac{\Psi}{R_D^3 \omega_p} \quad (3.96)$$

Specific energy

$$I_0 = \frac{I}{R_D^2 \omega_p^2} \quad (3.97)$$

Pressure

$$p_0 = \frac{p}{\rho R_D^2 \omega_p^2} \quad (3.98)$$

Force per unit mass

$$f_0 = \frac{f}{R_D \omega_p^2} \quad (3.99)$$

Vorticity

$$\xi_0 = \frac{\xi}{\omega_p} \quad (3.100)$$

Time

$$t_0 = \frac{t}{1/\omega_p} \quad (3.101)$$

Rothalpy

$$H_0 = \frac{H}{R_D^2 \omega_p^2} \quad (3.102)$$

Dissipation function

$$\Phi_0 = \frac{\Phi}{\omega_p} \quad (3.103)$$

Moment of momentum

$$L_0 = \frac{L}{\rho R_D^5 \omega_p^2} \quad (3.104)$$

Torque

$$M_0 = \frac{M}{\rho R_D^5 \omega_p^2} \quad (3.105)$$

Force

$$F_0 = \frac{F}{\rho R_D^4 \omega_p^2} \quad (3.106)$$

Momentum

$$J_0 = \frac{J}{\rho R_D^4 \omega_p^2} \quad (3.107)$$

Power flux

$$W_0 = \frac{W}{\rho R_D^5 \omega_p^3} \quad (3.108)$$

Power

$$E_0 = \frac{E}{\rho R_D^5 \omega_p^3} \quad (3.109)$$

Mass flow

$$\dot{m}_0 = \frac{\dot{m}}{\rho R_D^3 \omega_p} \quad (3.110)$$

Volume flow, see Eq. (3.22)

$$\dot{V}_0 = \frac{\dot{V}}{R_D^3 \omega_p} \quad (3.111)$$

As $\dot{V} = \dot{m}/\rho$ the non-dimensional quantities $\dot{m}_0$ and $\dot{V}_0$ are identical.

Either the dimensional or the non-dimensional quantities above can be used in the equations, so that the subscript 0, denoting non-dimensional quantity, is only used when distinction is necessary.

4. Flow at Turbine Entrance

4.1. Shock Losses at Turbine Entrance

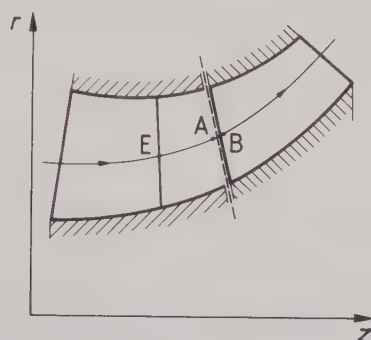


Fig. 4.1. Flow between two turbines

This chapter deals with the flow at the entrance of a turbine. In Fig. 4.1 a meridional streamline through two turbines is shown. At E the fluid leaves the blades in the first turbine and reaches its exit at A. At B, which is adjacent to A, the fluid has just entered the second turbine. When the fluid reaches the point A it has a certain circumferential velocity $c_{\theta A}$ and hence the relative circumferential velocity $w_{\theta AB} = c_{\theta A} - u_B$ from the horizon of the second tur-

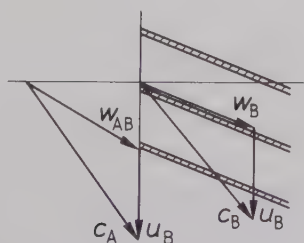


Fig. 4.2. Relative velocities at turbine entrance

bine. In general, $w_{\theta AB}$ does not fit the blades at B which implies that the circumferential velocity has to be changed abruptly on entering the turbine, see Fig. 4.2.

Fig. 4.3. b shows the flow at the entrance of a cascade. A control volume

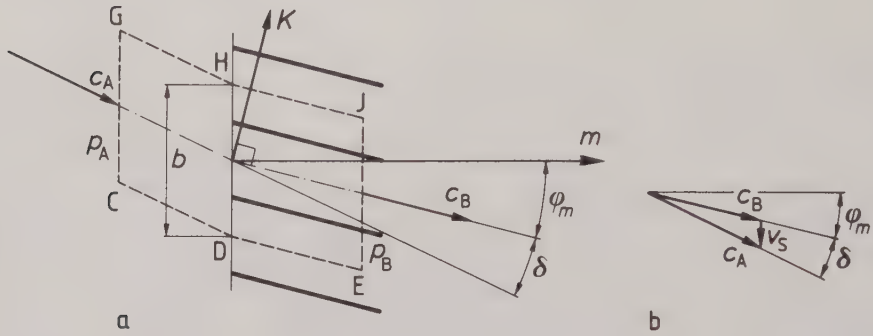


Fig. 4.3. Control volume at turbine entrance

CDEJHG is constructed so that the sides CDE and GHJ are streamlines and CG, DH, and EJ all have the length b while the height is h .

The fluid entering at CG is equal to the discharge at EJ, that is

$$\dot{m} = \rho c_{mA} b h = \rho c_{mB} b h \quad (4.1)$$

The flow is assumed to be non-dissipative and hence the force K caused by the blades is perpendicular to these. As the forces at the sides CDE and GHJ oppose each other, the equation of motion in the direction of c_B is

$$\dot{m}(c_B - c_A \cos \delta) = (p_A - p_B) b h \cos \varphi_m$$

Eq. (4.1) and Fig. 4.3. a give

$$c_{mA} = c_{mB} = c_B \cos \varphi_m = c_A \cos (\varphi_m + \delta) \quad (4.2)$$

giving the pressure difference

$$\frac{p_A - p_B}{\rho} = c_B (c_B - c_A \cos \delta) \quad (4.3)$$

The change in specific energy, Eq. (3.31), is

$$I_B - I_A = \frac{p_B - p_A}{\rho} + \frac{c_B^2 - c_A^2}{2} = - \frac{c_A^2 + c_B^2 - 2 c_A c_B \cos \delta}{2}$$

Introducing the shock velocity v_S as

$$\vec{v}_S = \vec{c}_A - \vec{c}_B \quad (4.4)$$

and applying the cosine theorem to this, see Fig. 4.3. b, gives

$$I_B - I_A = -\Delta I_S = - \frac{v_S^2}{2} \quad (4.5)$$

The sudden deflection of the velocity on entering the cascade thus causes an energy loss.

Usually this energy loss is written

$$\Delta I_S = \kappa \frac{v_S^2}{2} \quad (4.6)$$

where a shock factor κ is introduced. Several values are given for this, $0.5 \div 0.7$ in Ref. [26] and $0.5 \div 0.85$ in [5], but according to [4], [8], and [24] the value 1.0 is the most appropriate for torque converters.

4.2. Flow Properties at Turbine Entrance

As the turbine considered is assumed to have an infinite number of blades there is no room for any transition zone in it; the fluid must follow the blades exactly everywhere. The velocity of the fluid may thus be assumed to be abruptly changed just at the entrance of the turbine. The control volume in Fig. 4.3 can therefore be made very thin. An abrupt change of velocity can only be caused by discrete forces, not by forces with limited intensity, such as $\vec{f}$ and $\nabla p/\rho$. In Fig. 4.4 a thin control volume at the entrance surface is shown. Its front and back have both the area ΔA and the normal $\hat{a}$, see also Fig. 3.5

$$\hat{a} = (-\cos \alpha, 0, \sin \alpha) \quad (4.7)$$

The pressure is p_A at the front and p_B at the back, causing the discrete force

$$\vec{Q} = (p_A - p_B) \Delta A \hat{a} \quad (4.8)$$

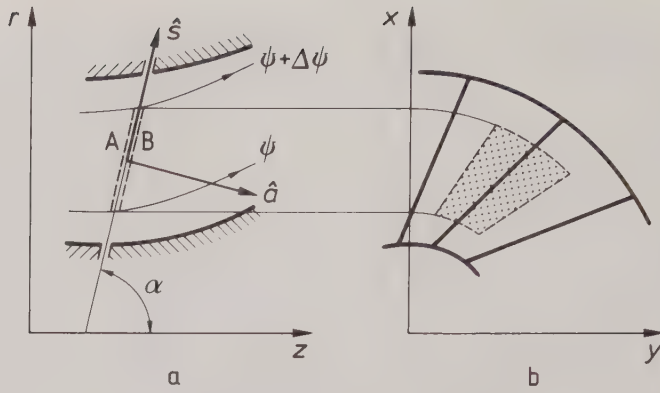


Fig. 4.4. Control volume at turbine entrance

As non-dissipative flow is considered, a force perpendicular to the blades acts at their inlet edges. According to Eq. (3.37) this force may be written

$$\vec{K} = K \hat{n} = K \vec{n} / n \quad (4.9)$$

The equation of motion for the control volume is

$$\vec{Q} + \vec{K} = \Delta \dot{m} (\vec{c}_B - \vec{c}_A) \quad (4.10)$$

where the flow of mass $\Delta \dot{m}$ through it is

$$\Delta \dot{m} = \rho \vec{c} \cdot \Delta \vec{A} = \rho \vec{c}_A \cdot \hat{a} \Delta A = \rho \vec{c}_B \cdot \hat{a} \Delta A \quad (4.11)$$

or, using Eqs. (3.8) and (3.20)

$$\frac{\Delta \dot{m}}{\rho \Delta A} = c_{aA} = c_{mA} \cos \lambda_A = c_{aB} = c_{mB} \cos \lambda_B = \frac{1}{r} \frac{\partial \Psi}{\partial s} \quad (4.12)$$

Forming the dot product between $\hat{a}$ and Eq. (4.10) yields

$$\vec{Q} \cdot \hat{a} + \vec{K} \cdot \hat{a} = \Delta \dot{m} (c_{aB} - c_{aA}) = 0 \quad (4.13)$$

implying that the total force is perpendicular to $\hat{a}$.

The velocities relative to the entered turbine are

$$\begin{cases} \vec{w}_B = \vec{c}_B - \vec{u}_B = \vec{c}_B - r \omega_B \hat{\theta} \\ \vec{w}_{AB} = \vec{c}_A - \vec{u}_B = \vec{c}_A - r \omega_B \hat{\theta} \end{cases} \quad (4.14)$$

giving the shock velocity v_S , Eq. (4.4)

$$\vec{v}_S = \vec{c}_A - \vec{c}_B = \vec{w}_{AB} - \vec{w}_B \quad (4.15)$$

which inserted in Eq. (4.10) yields

$$\vec{Q} + \vec{K} = -\Delta \dot{m} \vec{v}_S \quad (4.16)$$

Forming the dot product between this and $\vec{w}_B$ yields, with the aid of Eqs. (4.8), (4.14), (4.7), (4.11), (4.9), and (3.12)

$$\frac{p_A - p_B}{\rho} = -\vec{v}_S \cdot \vec{w}_B \quad (4.17)$$

Introduce a vector $\vec{e} = \vec{n} \times \hat{a}$ which is perpendicular to both $\vec{n}$ and $\hat{a}$. Insertion of Eqs. (3.11), (4.7), (3.9), and (3.4) gives

$$\vec{e} = \vec{n} \times \hat{a} = (\sin \alpha, \tan \varphi_s, \cos \alpha) \quad (4.18)$$

The vector $\hat{s}$ is defined in Fig. 4.4.a

$$\hat{s} = (\sin \alpha, 0, \cos \alpha) \quad (4.19)$$

and hence

$$\hat{s} \cdot \vec{n} = -\tan \varphi_s \quad (4.20)$$

The dot product $\vec{v}_S \cdot \vec{e}$ is zero according to Eqs. (4.16), (4.8), (4.9), and (4.18). Furthermore, $\vec{v}_S \cdot \hat{a}$ is also zero, Eqs. (4.18) and (4.15), and hence $\vec{v}_S$ is parallel to the vector $\hat{a} \times \vec{e}$, that is

$$\left\{ \begin{array}{l} \vec{v}_S = v_S \frac{\hat{a} \times \vec{e}}{|\hat{a} \times \vec{e}|} = v_S (-\hat{s} \sin \varphi_s + \hat{\theta} \cos \varphi_s) \\ v_{Ss} = -v_S \sin \varphi_s \\ v_{S\theta} = v_S \cos \varphi_s \end{array} \right. \quad (4.21)$$

Combination of Eqs. (3.12), (4.15), (4.20), and (3.11) yields

$$\vec{w}_{AB} \cdot \vec{n} = w_{ABn} = \vec{v}_S \cdot \vec{n} = -v_{Ss} \tan \varphi_s + v_{S\theta} = \frac{v_S}{\cos \varphi_s}$$

$\Rightarrow$

$$\left\{ \begin{array}{l} v_S = w_{ABn} \cos \varphi_s \\ v_{Ss} = -w_{ABn} \sin \varphi_s \cos \varphi_s \\ v_{S\theta} = w_{ABn} \cos^2 \varphi_s \end{array} \right. \quad (4.22)$$

The discontinuity in the specific moment of momentum, Eqs. (3.34) and (4.15), is

$$\Gamma_B - \Gamma_A = r (c_{\theta B} - c_{\theta A}) = -r v_{S\theta} \quad (4.23)$$

and in the specific energy, Eqs. (3.31), (4.17), and (4.14)

$$I_B - I_A = \frac{p_B - p_A}{\rho} + \frac{c_B^2 - c_A^2}{2} = -\frac{v_S^2}{2} - r \omega_B v_{S\theta} = -\frac{v_S^2}{2} + \omega_B (\Gamma_B - \Gamma_A) \quad (4.24)$$

and hence the discontinuity in the rothalpy, Eq. (3.39), becomes

$$H_B - H_A = I_B - I_A - \omega_B \Gamma_B + \omega_A \Gamma_A = -\frac{v_S^2}{2} + (\omega_A - \omega_B) \Gamma_A \quad (4.25)$$

Insertion of Eqs. (4.15), (3.8), and (4.22) in the dot product $\vec{v}_S \cdot \hat{s}$ yields

$$\tan \lambda_B - \tan \lambda_A = -\frac{v_{Ss}}{c_a} = \frac{w_{ABn} \sin \varphi_s \cos \varphi_s}{c_a} \quad (4.26)$$

Finally the quantity w_{ABn} has to be evaluated. Eqs. (3.2), (3.1), (3.11), (3.9), and (3.4) give

$$\begin{aligned} w_{ABn} &= (c_{mA} \sin \gamma_A, w_{AB\theta}, c_{mA} \cos \gamma_A) \cdot (-\tan \varphi_r, 1, -\tan \varphi_z) \\ &= w_{AB\theta} - c_{mA} \tan \varphi_{BA} \end{aligned} \quad (4.27)$$

The subscript BA refers to some quantity at B with respect to the flow direction at A. Using Eqs. (3.34) and (4.14) yields

$$\left\{ \begin{aligned} w_{ABn} &= \frac{\Gamma_A}{r} - r \omega_B - c_{mA} \tan \varphi_{BA} \\ w_{ABn} &= r (\omega_A - \omega_B) + c_{mA} (\tan \varphi_{mA} - \tan \varphi_{BA}) \\ w_{ABn} &= \frac{\Gamma_A - \Gamma_{BA}}{r} \end{aligned} \right. \quad (4.28)$$

At B the blade angles φ_s and φ_v in the s and v directions, see Fig. 3.4, are assumed to be known. Analogously to Eq. (3.7) the angle λ_v is defined as

$$\lambda_v = \beta - \left(\alpha - \frac{\pi}{2} \right) \quad (4.29)$$

The quantity $\tan \varphi_{BA}$ is now evaluated by Eqs. (3.9) and (3.5)

$$\tan \varphi_{BA} = \frac{\sin (\lambda_A - \lambda_v) \tan \varphi_s + \cos \lambda_A \tan \varphi_v}{\cos \lambda_v} \quad (4.30)$$

By Eqs. (3.34) and (3.8),

$$\Gamma_{BA} - \Gamma_v = r (c_{mA} \tan \varphi_{BA} - c_v \tan \varphi_v) = r c_a (\tan \lambda_A - \tan \lambda_v) \tan \varphi_s \quad (4.31)$$

At B the actual flow direction is λ_B and hence the actual moment of momentum is Γ_B while Γ_{BA} and Γ_v would be the actual values if the flow direction were λ_A and λ_v respectively. Exchange of λ_v to λ_B in Eq. (4.31) and insertion of (4.26) and (4.28) gives

$$\Gamma_{BA} - \Gamma_B = -(\Gamma_A - \Gamma_{BA}) \sin^2 \varphi_s \quad (4.32)$$

$\Rightarrow$

$$\frac{\Gamma_B - \Gamma_A}{\Gamma_{BA} - \Gamma_A} = \cos^2 \varphi_s \quad (4.33)$$

which show that the actual change in Γ is lower than it would be if φ_s were zero, that is the inclination of the blades reduces the change in Γ due to the shock.

The loss of specific energy, see Eq. (4.5), is

$$\Delta I_S = \frac{v_S^2}{2} = \frac{w_{ABn}^2 \cos^2 \varphi_s}{2} = \frac{(\Gamma_{BA} - \Gamma_A)^2 \cos^2 \varphi_s}{2 r^2} = \frac{(\Gamma_B - \Gamma_A)^2}{2 r^2 \cos^2 \varphi_s} \quad (4.34)$$

The dot product $\vec{k} \cdot \hat{s}$, according to Eqs. (3.37) and (4.20), is: $k_s = k_\theta \vec{n} \cdot \hat{s} = -k_\theta \tan \varphi_s$. Use of Eqs. (3.35), (3.34), (3.39), and (3.55) gives, for $\Phi = 0$

$$\zeta_{\theta,F} = \frac{1}{c_a} \left(\frac{\Gamma}{r^2} \frac{\partial \Gamma}{\partial s} - \frac{\partial I}{\partial s} - k_s \right) \quad (4.35)$$

and hence the change in this quantity is

$$\zeta_{\theta,F,B} - \zeta_{\theta,F,A} = \frac{1}{c_a} \left\{ \frac{1}{2r^2} \frac{\partial}{\partial s} (\Gamma_B^2 - \Gamma_A^2) - \frac{\partial}{\partial s} (I_B - I_A) - (k_{sB} - k_{sA}) \right\} \quad (4.36)$$

that is, even if $\Gamma_B = \Gamma_A$ and $I_B = I_A$ the vorticity may be discontinuous due to a discontinuity in k_s . This may occur for instance at E in Fig. 4.1 when the fluid leaves the blades and k_s disappears.

Insertion of Eqs. (3.7) and (3.8) in (3.65) yields

$$\zeta_{\theta, \Psi} = -\frac{1}{\cos^2 \lambda} \frac{\partial c_a}{\partial s} + c_a \left\{ \frac{K}{\cos^3 \lambda} - \frac{\partial}{\partial s} \left(\frac{1}{\cos^2 \lambda} \right) + \frac{\cos \alpha \tan \lambda - \sin \alpha \tan^2 \lambda}{r} \right\} \quad (4.37)$$

and hence the discontinuity in the vorticity may also be written

$$\begin{aligned} \zeta_{\theta, \Psi, B} - \zeta_{\theta, \Psi, A} = & c_a \left(\frac{K_B}{\cos^3 \lambda_B} - \frac{K_A}{\cos^3 \lambda_A} \right) \\ & - \frac{\partial c_a}{\partial s} (\tan^2 \lambda_B - \tan^2 \lambda_A) - c_a \frac{\partial}{\partial s} (\tan^2 \lambda_B - \tan^2 \lambda_A) \\ & + \frac{c_a \cos \alpha}{r} (\tan \lambda_B - \tan \lambda_A) - \frac{c_a \sin \alpha}{r} (\tan^2 \lambda_B - \tan^2 \lambda_A) \end{aligned} \quad (4.38)$$

showing that the discontinuity in the vorticity is connected with a discontinuity in the radius of curvature of the streamlines.

4.3. An Example of Turbine Entry Flow

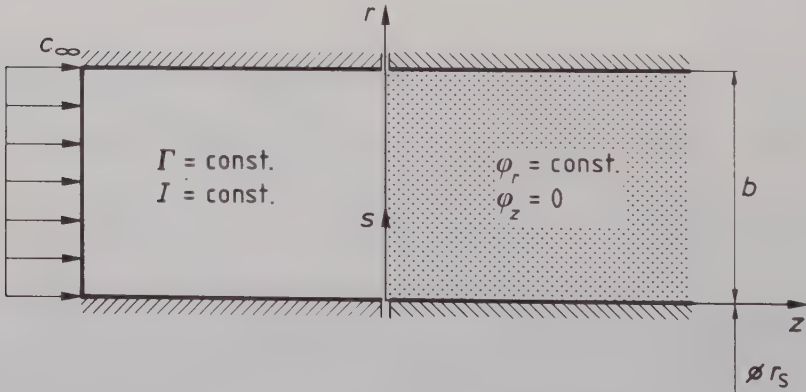


Fig. 4.5. Geometry and inlet flow for the turbine entry flow example

The flow in the neighbourhood of a turbine entrance may be further examined in a qualitative sense by constructing a simple example. Fig. 4.5 shows the meridional plane for an axial turbine occupying the interval $z \geq 0$ and having the blade angles $\varphi_r = \text{const.}$ and $\varphi_z = 0$. The space in front of it, $z < 0$, is empty. The fluid inlet is at $z \rightarrow -\infty$ where it has the properties: $c_r = 0$, $c_z = \text{const.}$, $\Gamma = \text{const.}$, $I = \text{const.}$. As $\alpha = \pi/2$, Eq. (3.3) gives $r = r_s + s$, where s is 0 and b at the respective walls. A further assumption is $b \ll r_s$, so henceforth r is treated as a constant and $\partial/\partial r$ replaced by $\partial/\partial s$.

As I and Γ are constant the vorticity ξ_θ , see Eq. (3.60), is zero and hence Eq. (3.64) is

$$\frac{\partial^2 \Psi}{\partial s^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0; \quad z < 0 \quad (4.39)$$

A solution to this is $\Psi = r c_\infty s$, which is applicable at $z = -\infty$ if $c_z(s, -\infty) = c_\infty$. Further boundary conditions are $\Psi(0, z) = 0$ and $\Psi(b, z) = r c_\infty b$. According to these the solution to the Laplace equation (4.39) is written

$$\Psi(s, z) = r c_\infty s + r \sum_{k=1}^{\infty} a_k \exp \frac{k \pi z}{b} \sin \frac{k \pi s}{b}; \quad z < 0 \quad (4.40)$$

giving the velocities, Eq. (3.18)

$$\begin{cases} c_s = - \sum_{k=1}^{\infty} a_k \frac{k \pi}{b} \exp \frac{k \pi z}{b} \sin \frac{k \pi s}{b} \\ c_z = \sum_{k=1}^{\infty} a_k \frac{k \pi}{b} \exp \frac{k \pi z}{b} \cos \frac{k \pi s}{b} + c_\infty \end{cases} \quad (4.41)$$

Inserting Eq. (4.30) in (4.28), and since $\lambda_v = 0$ and $\varphi_v = \varphi_z = 0$, yields

$$w_{ABn} = \frac{\Gamma_A}{r} - r \omega_B - c_{mA} \sin \lambda_A \tan \varphi_r$$

and using Eqs. (3.8) and (4.41)

$$w_{ABn} = \frac{\Gamma_A}{r} - r \omega_B + \tan \varphi_s \sum_{k=1}^{\infty} a_k \frac{k \pi}{b} \sin \frac{k \pi s_0}{b} \quad (4.42)$$

where s_0 is the s coordinate at the turbine entrance at $z = 0$.

Performing the derivations in Eqs. (3.64) and (3.61) yields in this special case

$$\zeta_\theta = -\frac{1}{r} \left(\frac{\partial^2 \Psi}{\partial s^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) = r \left(\frac{\partial \Theta}{\partial s} \right)^2 \frac{\partial^2 \Psi}{\partial z^2} - r \frac{dH_B}{d\Psi}; \quad z > 0$$

and with Eq. (3.9)

$$\frac{\partial^2 \Psi}{\partial s^2} + \frac{1}{\cos^2 \varphi_s} \frac{\partial^2 \Psi}{\partial z^2} = r^2 \frac{dH_B}{d\Psi}; \quad z > 0 \quad (4.43)$$

The rothalpy H_B after the shock is determined by Eq. (4.25) and its derivative is

$$\frac{dH_B}{d\Psi} = -\frac{d}{d\Psi} \left(\frac{v_S^2}{2} \right) = -\frac{d}{d\Psi} \Delta I_S \quad (4.44)$$

If this quantity is assumed to be small the solution to Eq. (4.43) is

$$\Psi(s, z) = r c_\infty s + r \sum_{k=1}^{\infty} b_k \exp \left(\frac{-k\pi \cos \varphi_s z}{b} \right) \sin \frac{k\pi s}{b}; \quad z > 0 \quad (4.45)$$

and hence the velocities

$$\begin{cases} c_s = \cos \varphi_s \sum_{k=1}^{\infty} b_k \frac{k\pi}{b} \exp \left(\frac{-k\pi \cos \varphi_s z}{b} \right) \sin \frac{k\pi s}{b} \\ c_z = c_\infty + \sum_{k=1}^{\infty} b_k \frac{k\pi}{b} \exp \left(\frac{-k\pi \cos \varphi_s z}{b} \right) \cos \frac{k\pi s}{b} \end{cases} \quad (4.46)$$

The s component of the shock velocity, v_{Ss} , see Eqs. (4.15) and (3.8), is

$$v_{Ss} = c_{sA} - c_{sB} = c_a (\tan \lambda_A - \tan \lambda_B) = -\sum_{k=1}^{\infty} (a_k + b_k \cos \varphi_s) \frac{k\pi}{b} \sin \frac{k\pi s_0}{b} \quad (4.47)$$

The number 1 can be expressed as a Fourier series

$$\begin{cases} 1 = \sum_{k=1}^{\infty} \iota_k \sin \frac{k \pi s}{b} \\ \iota_k = \frac{2}{b} \int_0^b \sin \frac{k \pi s}{b} ds = \frac{2 (1 - (-1)^k)}{k \pi} \end{cases} \quad (4.48)$$

Insertion in Eq. (4.26) gives

$$(a_k + b_k \cos \varphi_s) \frac{k \pi}{b} = \sin \varphi_s \cos \varphi_s \left(\left(\frac{\Gamma_A}{r} - r \omega_B \right) \iota_k + a_k \frac{k \pi}{b} \tan \varphi_s \right) \quad (4.49)$$

Furthermore the function Ψ is continuous, that is $\Psi(s, 0-) = \Psi(s, 0+)$, and hence $a_k = b_k$, see Eqs. (4.40) and (4.45). This inserted in Eq. (4.49) gives

$$a_k = b_k = \left(\frac{\Gamma_A}{r} - r \omega_B \right) \frac{b \iota_k}{k \pi} \tan \frac{\varphi_s}{2} \quad (4.50)$$

and with Eqs. (4.42) and (4.48)

$$\begin{aligned} w_{ABn} &= \left(\frac{\Gamma_A}{r} - r \omega_B \right) \left(1 + \tan \varphi_s \tan \frac{\varphi_s}{2} \sum_{k=1}^{\infty} \iota_k \sin \frac{k \pi s_0}{b} \right) = \\ &= \frac{1}{\cos \varphi_s} \left(\frac{\Gamma_A}{r} - r \omega_B \right) \end{aligned} \quad (4.51)$$

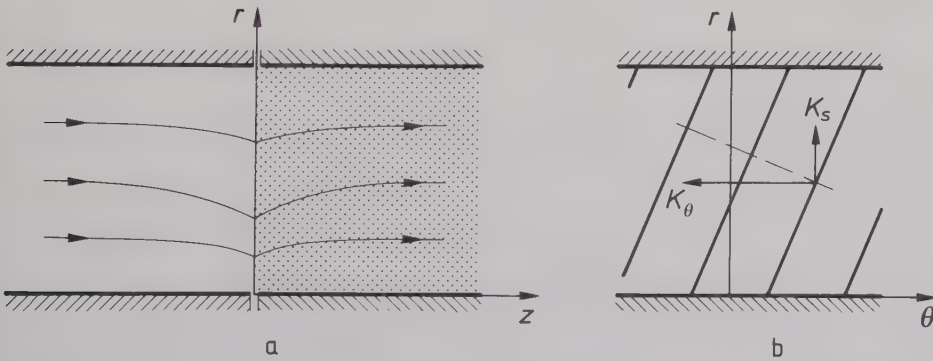


Fig. 4.6. Streamlines a) and blade forces b) shown schematically for $\Gamma_A > r^2 \omega_B$ and $\varphi_s > 0$

and finally the specific energy loss becomes

$$\Delta I_S = \frac{1}{2} \left(\frac{\Gamma_A}{r} - r \omega_B \right)^2 \quad (4.52)$$

Thus it turns out that this is independent of Ψ and hence the assumption that $dH_B/d\Psi$ is small is quite correct. Furthermore ΔI_S is independent of the angle φ_s .

Thus, at the turbine entrance is

$$\left\{ \begin{array}{l} \Psi_0 = r c_\infty s + (\Gamma_A - r^2 \omega_B) \tan \frac{\varphi_s}{2} \sum_{k=1}^{\infty} \frac{b}{k \pi} \sin \frac{k \pi s_0}{b} \\ c_{sA} = - \left(\frac{\Gamma_A}{r} - r \omega_B \right) \tan \frac{\varphi_s}{2} \\ c_{sB} = \left(\frac{\Gamma_A}{r} - r \omega_B \right) \tan \frac{\varphi_s}{2} \cos \varphi_s \\ c_{z0} = c_\infty + \left(\frac{\Gamma_A}{r} - r \omega_B \right) \tan \frac{\varphi_s}{2} \sum_{k=1}^{\infty} \iota_k \cos \frac{k \pi s_0}{b} \\ v_{Ss} = - \left(\frac{\Gamma_A}{r} - r \omega_B \right) \sin \varphi_s \\ v_{S\theta} = \left(\frac{\Gamma_A}{r} - r \omega_B \right) \cos \varphi_s \\ \Gamma_B = \Gamma_A - (\Gamma_A - r^2 \omega_B) \cos \varphi_s \end{array} \right. \quad (4.53)$$

When $z \rightarrow \infty$ the stream function returns to $r c_\infty s$ at a rate depending on $\exp(-\pi \cos \varphi_s z/b)$.

Fig. 4.6 shows the streamlines and the actual directions of the blade forces for $\Gamma_A > r^2 \omega_B$ and $\varphi_s > 0$.

5. The Streamline Curvature Method

5.1. The Algorithms

A method frequently used to solve the flow equations is the Streamline Curvature Method, (SCM). By this method a number of streamlines are traced through the turbine. Examples of its use are given in Refs. [14], [15], and [17].

In the (i, k) system, see Fig. 3.4, the k -lines are specified at the beginning of the calculation while the i -lines are streamlines. The transformation $(r, z) \rightarrow (i, k)$ is assumed to give integer values for i and k on the i - and k -lines considered, but does not have to be expressed explicitly; only certain geometrical properties along the i - and k -lines are needed. All quantities are known or evaluated only at the intersection of these i - and k -lines.

The calculation starts by choosing a set of initial i -lines, that is s is chosen. Then r and z are calculated by means of Eq. (3.3). A curve in the r - z plane may be written as $\vec{r} = (r(\tau), 0, z(\tau))$ where τ is a parameter. The length between the

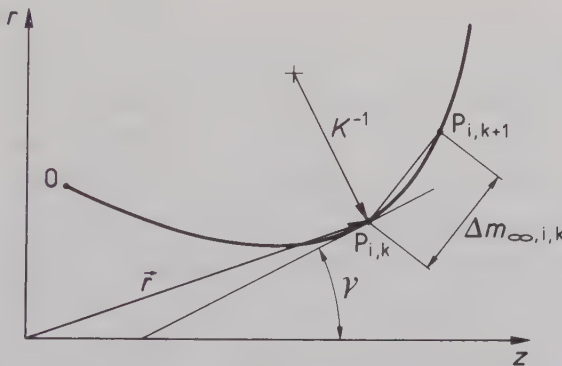


Fig. 5.1. Geometry of a meridional streamline

adjacent points $P_{i,k}$ and $P_{i,k+1}$ on an i -line, see Fig. 5.1, is

$$\Delta m_{\infty, i, k} = |\vec{r}_i(\tau_{k+1}) - \vec{r}_i(\tau_k)| = \sqrt{(r_{i,k+1} - r_{i,k})^2 + (z_{i,k+1} - z_{i,k})^2} \quad (5.1)$$

Adding these lengths yields the total length m along the streamline from some point O.

The angle γ and the curvature K , see Fig. 5.1, are

$$\tan \gamma = \frac{r'}{z'} \quad (5.2)$$

$$K = (r'' z' - z'' r') (r'^2 + z'^2)^{-3/2} \quad (5.3)$$

giving the angle λ , Eq. (3.7)

$$\tan \lambda = \cot(\alpha - \gamma) = \frac{z' \cos \alpha + r' \sin \alpha}{z' \sin \alpha - r' \cos \alpha} \quad (5.4)$$

where the prime denotes the derivative with respect to τ .

The value Δm_∞ in Eq. (5.1) will always be less than the actual value Δm . If the actual curve were a circle with radius R , Δm would be: $2R \arcsin(\Delta m_\infty/2R)$. As $\arcsin x \approx x + x^3/6$ an improved alternative to Eq. (5.1) is

$$\Delta m_{i,k} = \Delta m_{\infty i,k} + \frac{1}{48} \Delta m_{\infty i,k}^3 (K_{i,k}^2 + K_{i,k+1}^2) \quad (5.5)$$

The correction due to the curvature is usually small. The calculation of the geometry is thus carried out in the order: r , z , m , λ , and K , and m is used as parameter, that is: $\tau = m$. The blade angles φ_s and φ_m are evaluated with the aid of Eq. (3.9); the values for Θ are indata.

Furthermore the rothalpy H and the stream function Ψ must be known somewhere, for instance at the inlet of the turbine. Ψ is constant along a streamline while H is declining in accordance with the dissipation function Φ , see Eq. (3.42). If the flow is non-dissipative, that is $\Phi \equiv 0$, the rothalpy is also constant along a streamline. Any quantity desired may now be evaluated with the aid of the equations in Chap. 3.

These calculations imply numerical differentiations. The function considered, say $q(x)$, is known at a number of points, $q(x_i) = q_i$, and may be approximated by the Lagrangian interpolation polynomial

$$Q(x) = \sum_{j=1}^n q_j \left(\prod_{\substack{k=1 \\ k \neq j}}^n (x - x_k) \right) \left(\prod_{\substack{k=1 \\ k \neq j}}^n (x_j - x_k) \right)^{-1} \quad (5.6)$$

The derivative $q'(x_i)$ is approximately $Q'(x_i)$. Throughout the calculations n is 3, that is $Q(x)$ is a parabola, and the three points $j = 1, 2, 3$ are those which

are closest to the actual $\alpha = \alpha_i$.

The polynomial is also used to perform numerical integrations.

Define a quantity f as

$$f = - \left(\frac{\partial c_m}{\partial s} + A c_m + B + \frac{C}{c_m} \right) \quad (5.7)$$

where A , B , and C are given in Eq. (3.66). The f is calculated using the now accessible values and its deviation from zero indicates to what extent the Eq. (3.66) is satisfied. If the streamlines are displaced f may be lessened. Introduce the corrective velocity ϵ in such a way that if $\partial c_m / \partial s$ were replaced by $\partial(c_m + \epsilon) / \partial s$ Eq. (3.66) would be satisfied. Solving this for ϵ gives then

$$\frac{\partial \epsilon}{\partial s} = f \quad (5.8)$$

An alternative is to solve Eq. (3.66) with respect to c_m , but the coefficients A , B , C are dependent on the geometry of the streamlines. The solution procedure is therefore iterative anyhow, thus justifying the method used.

Analogously to Eq. (3.19) a corrective stream function Ψ_ϵ is defined as

$$\frac{\partial \Psi_\epsilon}{\partial s} = r \epsilon \cos \lambda \quad (5.9)$$

and a new stream function is

$$\Psi^{(n+1)} = \Psi^{(n)} + \Lambda \Psi_\epsilon \quad (5.10)$$

where a damping coefficient Λ is introduced to control the extent to which the obtained correction should be done.

As the total through flow $\dot{m}$ is known, Ψ is also known at the boundaries, see Eq. (3.21), hence implying the boundary conditions for Ψ_ϵ

$$\Psi_{\epsilon,CD} = \Psi_{\epsilon,AB} = 0 \quad (5.11)$$

The corrective stream function is obtained by applying the trapezoidal formula on Eq. (5.9)

$$\Psi_{\epsilon,k} = \int_{s_1}^{s_k} \epsilon r \cos \lambda ds = \sum_{j=1}^{k-1} \frac{1}{2} \left((\epsilon r \cos \lambda)_j + (\epsilon r \cos \lambda)_{j+1} \right) (s_{j+1} - s_j) \quad (5.12)$$

If the number of traced streamlines is T , Eq. (5.8) yields $T - 1$ differences

$$\epsilon_{j+1} - \epsilon_j = \frac{1}{2} (f_{j+1} + f_j) (s_{j+1} - s_j) \quad (5.13)$$

An additional condition is $\psi_{\epsilon, T} = 0$ which is expressed by Eq. (5.12). These T equations are solved with respect to the corrective velocities and then Ψ_{ϵ} is obtainable.

The final step is to displace the streamlines. The new stream function $\Psi(s)$ is obtained by means of Eq. (5.10). A streamline is characterized by its stream

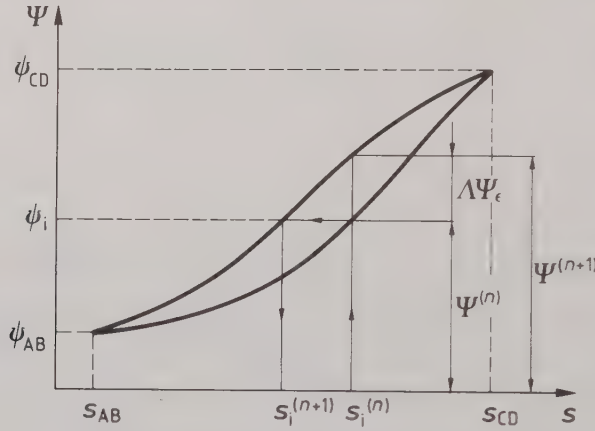


Fig. 5.2. Change in streamline position due to the change of the stream function

function value ψ_i and hence s is obtained from

$$s_i = \Psi^{-1}(\psi_i) \quad (5.14)$$

This is calculated by inverse interpolation, see Fig. 5.2, in which the formula (5.6) is utilized. An approximative expression for the new streamline position is obtained from Fig. 5.2.

$$s_i^{(n+1)} \approx s_i^{(n)} - \Lambda \Psi_{\epsilon} \left(\frac{\partial \Psi}{\partial s} \right)^{-1} \quad (5.15)$$

At the entrances of the turbines two k-lines are located in the same position; one line considers the flow just before the entry shock, A, and the other just afterwards, B. At these two lines the solution procedure should produce identical results as the stream function is continuous. This would not occur in practice due to the errors introduced by the numerical differentiations. The positions

of the streamlines are therefore determined with f as $(f_A + f_B)/2$.

Furthermore the values produced at A and B can not be expected to satisfy Eq. (4.26) with (4.28) and (4.30) inserted. Instead a mean flow angle λ^* is introduced

$$\tan \lambda^* = \frac{1}{2} (\tan \lambda_A + \tan \lambda_B) \quad (5.16)$$

and the corresponding specific moment of momentum Γ^* , using Eq. (4.31) where the A and v directions are exchanged for the $*$ and B directions, is

$$\Gamma^* - \Gamma_B = r c_a (\tan \lambda^* - \tan \lambda_B) \tan \varphi_s \quad (5.17)$$

Insertion of Eqs. (5.16) and (4.26) yields

$$\Gamma^* - \Gamma_B = -\frac{r w_{ABn} \sin^2 \varphi_s}{2}$$

Further insertion of Eqs. (4.28) and (4.33) gives

$$w_{ABn} = \frac{2}{1 + \cos^2 \varphi_s} \frac{\Gamma_A - \Gamma^*}{r} \quad (5.18)$$

which inserted in Eq. (4.34) gives the specific energy loss

$$\left\{ \begin{aligned} \Delta I_S &= \frac{\kappa^*}{2} \left(\frac{\Gamma_A - \Gamma^*}{r} \right)^2 \\ \kappa^* &= \frac{4 \cos^2 \varphi_s}{(1 + \cos^2 \varphi_s)^2} = \frac{1}{1 + \frac{\tan^4 \varphi_s}{4(1 + \tan^2 \varphi_s)}} \end{aligned} \right. \quad (5.19)$$

Notice that Γ^* can be calculated using any (λ, Γ) pair, not necessarily the actual one in the B direction; compare with Eq. (4.31). Furthermore the value for λ^* may be more accurate than those for λ_A and λ_B , yielding a more accurate value for the specific energy loss. A final remark is that κ^* does not differ much from 1 for the usual values for φ_s , for instance $\kappa^*(25^\circ) = 0.990$ and $\kappa^*(50^\circ) = 0.828$.

With Eqs. (5.19) and (4.25) the rothalpy changes between the turbines are calculated. The rothalpy have to be known only at one position on each streamline, for instance at the inlet of the first turbine.

These calculations are carried out k-line by k-line through the entire row of turbines over and over until sufficiently small values are obtained for the

corrective stream function.

5.2. Qualitative Analysis of the Convergence

The working of the SCM can be studied in a qualitative sense by considering a simplified case. Any quantity may be written as a sum of the correct value and a deviation from this, that is: $A = A_C + \Delta A$. Applying this to Eq. (5.7) and noticing that the correct values satisfy (3.66) gives, if products of deviation quantities are neglected,

$$f = - \left(\frac{\partial \Delta c_m}{\partial s} + A_C \Delta c_m + \Delta A c_{m,C} + \Delta B + \Delta \left(\frac{C}{c_m} \right) \right) \quad (5.20)$$

Assume furthermore that the flow angle λ is approximately zero and that r , φ_m , and φ_s do not change much due to the deviations of the streamlines. Finally the flow is assumed to be non-dissipative and hence Eq. (3.58) applies. The deviations in the A , B , C 's become now

$$\left\{ \begin{array}{l} \Delta A = -\cos^2 \varphi_m \Delta K \\ \Delta B = 0 \\ \Delta \left(\frac{C}{c_m} \right) = -r \cos^2 \varphi_m \Delta \left(\frac{dH}{d\Psi} \right) \end{array} \right. \quad (5.21)$$

The function $H'(\Psi)$, prime denote derivative with respect to Ψ , is known but at a certain s coordinate a distorted Ψ is inserted in it causing a deviation in $H'(\Psi(s))$ which is: $\Delta H'(\Psi(s)) \approx H_C'' \Delta \Psi$.

In Fig. 5.3 a local coordinate system (τ, s) is fitted. Now the curvature may be written approximately as $d^2 s / d\tau^2$ and hence the deviation of the curvature is

$$\Delta K = \frac{d^2 \Delta s}{d\tau^2} \quad (5.22)$$

The curvature, however, is evaluated by a numerical method. Application of Eq. (5.6) here gives: $q_k'' = (q_{k+1} - 2q_k + q_{k-1})/h^2$. If only the streamline positions at the considered k -line are assumed to be distorted, the deviation of the curvature may be written

$$\Delta K = -\frac{e_D}{h^2} \Delta s = \frac{\partial q_k''}{\partial q_k} \Delta s \quad (5.23)$$

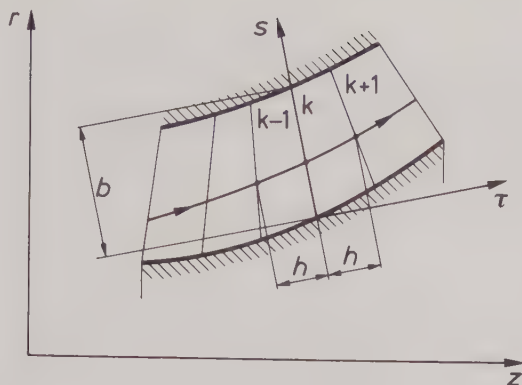


Fig. 5.3. Local coordinate system at a k-line

where e_D is a constant depending on the numerical method used; in the method mentioned it is 2.

Eq. (5.15) gives a connection between the deviations of the streamline position and the deviation of the streamline function which is also connected to the deviation of the velocity by Eq. (3.19)

$$\Delta s = -\frac{\Delta \Psi}{r c_{m,C}} \quad (5.24)$$

$$\Delta c_m = \frac{1}{r} \frac{d \Delta \Psi}{ds} \quad (5.25)$$

Insertion in Eq. (5.20) yields

$$f = -\left\{ \frac{1}{r} \frac{d^2 \Delta \Psi}{ds^2} + \frac{A_C}{r} \frac{d \Delta \Psi}{ds} - \frac{\cos^2 \varphi_m}{r} \left(\frac{e_D}{h^2} + r^2 H_C'' \right) \Delta \Psi \right\} \quad (5.26)$$

As the stream function is known at $s = 0$ and $s = b$, see Fig. 5.3, that is, there is no deviation at the walls and hence $\Delta \Psi$ may be expressed as a sine series,

$$\Delta \Psi = \sum_{j=1}^{\infty} e_j \sin \frac{j\pi s}{b} \quad (5.27)$$

The corrective stream function Ψ_ϵ is evaluated by Eqs. (5.8), (5.9), (5.26), and (5.27)

$$\frac{\partial^2 \Psi_\epsilon}{\partial s^2} = \sum_{j=1}^{\infty} \left\{ \left(\frac{j\pi}{b} \right)^2 + \cos^2 \varphi_m \left(\frac{e_D}{h^2} + r^2 H_C'' \right) \right\} \sin \frac{j\pi s}{b} + \frac{j\pi}{b} A_C \cos \frac{j\pi s}{b} \left\} e_j \quad (5.28)$$

If the A_C term is omitted a simple solution to this emerges, namely

$$\Psi_\epsilon = - \sum_{j=1}^{\infty} e_j \left\{ 1 + \left(\frac{b \cos \varphi_m}{j\pi} \right)^2 \left(\frac{e_D}{h^2} + r^2 H_C'' \right) \right\} \sin \frac{j\pi s}{b} \quad (5.29)$$

A new stream function is obtained by Eq. (5.10) and its deviation is

$$\Delta \Psi^{(n+1)} = \sum_{j=1}^{\infty} e_j^{(n)} \left\{ 1 - \Lambda \left\{ 1 + \left(\frac{b \cos \varphi_m}{j\pi} \right)^2 \left(\frac{e_D}{h^2} + r^2 H_C'' \right) \right\} \right\} \sin \frac{j\pi s}{b} \quad (5.30)$$

that is a new sine series with the coefficients $e_j^{(n+1)}$. If the first one is assumed to be dominant, the best choice of the damping coefficient is

$$\Lambda_0 = \frac{1}{1 + \left(\frac{b \cos \varphi_m}{\pi} \right)^2 \left(\frac{e_D}{h^2} + r^2 H_C'' \right)} \quad (5.31)$$

However e_D is not 2 everywhere; for instance $e_D = -1$ at an inlet as the second derivative here is evaluated as $q_k'' = (q_k - 2q_{k+1} + q_{k+2})/h^2$, see Eq. (5.23). If furthermore $\varphi_m = 0$ and $H_C'' = 0$, the new deviations are

$$\frac{e_1^{(n+1)}}{e_1^{(n)}} = \begin{cases} 1 - \Lambda \left(1 - \left(\frac{b}{\pi h} \right)^2 \right) & \text{inlet/outlet} \\ 1 - \Lambda \left(1 + 2 \left(\frac{b}{\pi h} \right)^2 \right) & \text{interior} \end{cases} \quad (5.32)$$

that is the deviations decline slower at the boundaries than in the interior. In fact if $h/b < 1/\pi$ the deviation at the boundaries will grow; that is no solution can be obtained. This phenomenon depends on the method employed when calculating the curvature. If, for instance, double differentiation is used instead, the following applies: $q_k'' = (q_{k-2} - 2q_k + q_{k+2})/4h^2$ that is, the e_D 's are reduced to a fourth, which unfortunately also applies for the accuracy of the values of the curvature.

In the same way it may be concluded that too high negative values of H_C'' may cause divergence.

The convergence of the SCM is dealt with by Ref. [25] where various methods of calculating the curvature are investigated.

The main drawback of the SCM is the difficulty in increasing the accuracy, that is in decreasing h , without destroying the convergence.

6. A Finite Element Method

6.1. Application of the Galerkin Method

Another way to solve the flow equations is to employ a finite element method, (FEM). The governing equations for Ψ are (3.64), (3.54), and (3.57).

$$\left\{ \begin{array}{l} -\left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \Psi}{\partial z} \right) \right) + g = 0 \\ g = -\zeta_{\theta, F} = \begin{cases} -\chi[\Gamma, \Theta] + \frac{1}{c_m \cos \lambda} \frac{\partial H}{\partial s} + \Phi \frac{\sin \lambda + \tan \varphi_s \tan \varphi_m}{\cos \lambda} \\ -\frac{\Gamma - r^2 \omega}{r^2 c_m \cos \lambda} \frac{\partial \Gamma}{\partial s} + \frac{1}{c_m \cos \lambda} \frac{\partial H}{\partial s} + \Phi \tan \lambda \quad (\text{no blades}) \end{cases} \end{array} \right. \quad (6.1)$$

The problem is thus to solve an equation of Poisson likeness in a domain D with the bounding curve C , the direction of which is determined by $\hat{\theta} = \hat{z} \times \hat{r}$, see Fig. 6.1. As the stream function is involved in g this method is iterative too.

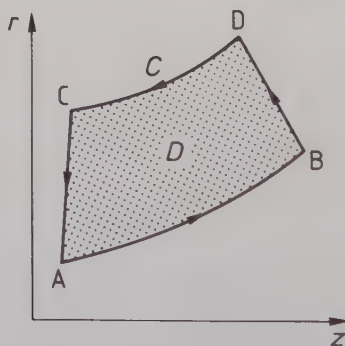


Fig. 6.1. The meridional domain D for a turbine and its bounding curve C

The stream function is expressed with the aid of a set of base functions $M_j(r, z)$ which fulfil

$$M_j(r_i, z_i) = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases} \quad (6.2)$$

The stream function can now be expressed as

$$\Psi(r, z) = \sum_{j=1}^m \psi_j M_j(r, z) \quad (6.3)$$

that is, the function is described by m constants ψ_j which are the values of the function in the positions (r_j, z_j) , also referred to as the nodes. The problem of finding the unknown function is thus converted to finding a number of nodal values.

Insertion of the approximative expression for the stream function (6.3) in (6.1) yields in general a residue e . For the exact solution e would be zero everywhere, but otherwise it is only achievable to make e zero in an average sense. By means of the Galerkin Method this is formulated as

$$\iint_D e M_k dz dr = 0 ; k = 1, \dots, m \quad (6.4)$$

The base functions M_k serve as weight functions. If M_k is assumed to be significant only near the node k , Eq. (6.4) implies that the residue becomes small around every node. Eq. (6.4) provides m conditions determining the m nodal stream function values.

In Refs. [1], [2], [9], and [11] examples of the application of this concept are shown.

The Stokes' theorem reads

$$\iint_D \left(\frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} \right) dz dr = \oint_C (-v_r dz + v_z dr)$$

and yields the formula

$$\iint_D \left(M_k \frac{\partial v_z}{\partial z} + M_k \frac{\partial v_r}{\partial r} \right) dz dr + \iint_D \left(v_z \frac{\partial M_k}{\partial z} + v_r \frac{\partial M_k}{\partial r} \right) dz dr = \oint_C M_k (v_z dr - v_r dz) \quad (6.5)$$

This is utilized when evaluating Eq. (6.4) with e as (6.1).

$$\iint_D \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \frac{\partial M_k}{\partial r} + \frac{1}{r} \frac{\partial \Psi}{\partial z} \frac{\partial M_k}{\partial z} \right) dz dr + \iint_D g M_k dz dr + \oint_C q M_k ds = 0 \quad (6.6)$$

where the contour integral has been developed, using $dr = \sin \alpha ds$, $dz = \cos \alpha ds$ and Eqs. (3.18) and (3.8), to

$$-\oint_C M_k \left(\frac{1}{r} \frac{\partial \Psi}{\partial z} dr - \frac{1}{r} \frac{\partial \Psi}{\partial r} dz \right) = \oint_C (c_r \sin \alpha + c_z \cos \alpha) M_k ds = \oint_C c_s M_k ds$$

that is

$$q = c_s \quad (6.7)$$

This partial integration can also be performed for the χ quantity involved in g .

$$\begin{aligned} -\iint_D M_k \chi[\Gamma, \Theta] dz dr &= \iint_D M_k \left\{ \frac{\partial}{\partial z} \left(\Gamma \frac{\partial \Theta}{\partial r} \right) - \frac{\partial}{\partial r} \left(\Gamma \frac{\partial \Theta}{\partial z} \right) \right\} dz dr = \\ &= \iint_D \Gamma \left(\frac{\partial \Theta}{\partial z} \frac{\partial M_k}{\partial r} - \frac{\partial \Theta}{\partial r} \frac{\partial M_k}{\partial z} \right) dz dr + \oint_C q_\chi M_k ds \end{aligned} \quad (6.8)$$

and recalling Eqs. (3.4) and (3.9)

$$q_\chi = \Gamma \frac{\partial \Theta}{\partial r} \sin \alpha + \Gamma \frac{\partial \Theta}{\partial z} \cos \alpha = \Gamma \frac{\partial \Theta}{\partial s} = \frac{\Gamma \tan \varphi_s}{r} \quad (6.9)$$

The bounding curve C is divided into four parts, see Fig. 6.1.

$$C = C_{AB} + C_{BD} + C_{DC} + C_{CA} \quad (6.10)$$

If a row of turbines are considered and if the j 'th turbine occupies the domain D_j and has the bounding curve C_j , $C_j = \partial D_j$, these quantities for the row are

$$\begin{cases} D = \bigcup_j D_j \\ C = \partial D + \bigcup_j (C_{BD,j} + C_{CA,j+1}) \end{cases} \quad (6.11)$$

Thus internal borderlines exist between the turbines and for two adjacent turbines $C_{AC, j+1} = C_{BD, j}$. The contour integrals involve only nodes on the boundaries. If a nodal value is known its corresponding Eq. (6.6) is not needed.

Usually the stream function is known at the boundary ∂D and hence the contour integrals only have to be considered at the internal borderlines. The contour integrals here yield

$$\int_{C_{BD,j}} q_{BD,j} ds + \int_{-C_{AC,j+1}} q_{AC,j+1} ds = \int_{C_{BD,j}} (q_A - q_B) ds \quad (6.12)$$

The subscripts A and B refer to quantities before and after shock respectively. Eqs. (6.7) and (4.15) give

$$\begin{cases} \Delta q = q_A - q_B = c_{sA} - c_{sB} = v_{Ss} \\ \Delta q_x = (\Gamma_A \tan \varphi_{sA} - \Gamma_B \tan \varphi_{sB})/r \end{cases} \quad (6.13)$$

The internal borderlines exist at the inlets and outlets of the turbines. Shocks occur only at inlets and hence Δq exist only there but Δq_x exist whenever φ_s is changed discontinuously. A space without blades is treated as one with $\varphi_s \equiv 0$.

Eq. (6.13) shows that Δq_x is often considerably greater than Δq . If the specific moment of momentum Γ is almost constant the partial integration (6.8) implies that a small expression is expanded into a difference of two expressions which may be great. The numerical evaluation of these may introduce significant errors and therefore the X term is kept in g .

Insertion of Eq. (6.3) in (6.6) and application of (6.3) to g and Δq too yields

$$\begin{aligned} \sum_{j=1}^m \psi_j \iint_D \left(\frac{\partial M_j}{\partial r} \frac{\partial M_k}{\partial r} + \frac{\partial M_j}{\partial z} \frac{\partial M_k}{\partial z} \right) \frac{dz dr}{r} + \\ \sum_{j=1}^m g_j \iint_D M_j M_k dz dr + \sum_{j=1}^m \Delta q_j \oint_C M_j M_k ds = 0 \\ k = 1, \dots, m \end{aligned} \quad (6.14)$$

In the last sum Δq is exchanged for c_s , Eq. (6.7), for nodes on the outer boundary while Eq. (6.13) applies for the internal borderlines. Collection of the nodal values in m -dimensional vectors allows Eq. (6.14) to be written in matrix form

$$\left\{ \begin{array}{l} \mathbf{K} \vec{\psi} + \mathbf{H} \vec{g} + \mathbf{D} \vec{\Delta q} = 0 \\ k_{kj} = \iint_D \left(\frac{\partial M_k}{\partial r} \frac{\partial M_j}{\partial r} + \frac{\partial M_k}{\partial z} \frac{\partial M_j}{\partial z} \right) \frac{dz dr}{r} \\ h_{kj} = \iint_D M_k M_j dz dr \\ d_{kj} = \oint_C M_k M_j ds \end{array} \right. \quad (6.15)$$

The d_{kj} 's are defined only for the nodes on the boundaries C .

6.2. A Variational Formulation

Some partial differential equations may be solved by employing the calculus of variations, Ref. [6]. Consider the functional

$$\Xi[\Psi] = \iint_D \left\{ \frac{1}{2r} \left(\left(\frac{\partial \Psi}{\partial r} \right)^2 + \left(\frac{\partial \Psi}{\partial z} \right)^2 \right) + g \Psi \right\} dz dr + \oint_C \sigma \Psi ds \quad (6.16)$$

and investigate how it reacts if Ψ is changed by η

$$\Xi[\Psi + \eta] - \Xi[\Psi] = \Xi[\eta] + \iint_D \left(\frac{\partial \Psi}{\partial r} \frac{\partial \eta}{\partial r} + \frac{\partial \Psi}{\partial z} \frac{\partial \eta}{\partial z} \right) \frac{dz dr}{r}$$

and apply Eq. (6.5) and the derivation which led to (6.7)

$$\begin{aligned} \Xi[\Psi + \eta] - \Xi[\Psi] &= \iint_D \frac{1}{2r} \left(\left(\frac{\partial \eta}{\partial r} \right)^2 + \left(\frac{\partial \eta}{\partial z} \right)^2 \right) dz dr \\ &+ \iint_D \left\{ g - \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \Psi}{\partial z} \right) \right) \right\} \eta dz dr + \oint_C (\sigma - c_s) \eta ds \end{aligned} \quad (6.17)$$

For those parts of the boundary where Ψ is prescribed $\eta = 0$ throughout the variation, that is the boundary condition is fulfilled always. Thus the contour integral only has to be considered on the remaining parts of the boundary.

Assume that

$$\begin{cases} -\left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r}\right) + \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \Psi}{\partial z}\right)\right) + g = 0 \text{ in } D \\ c_s = \sigma \text{ on } C \end{cases} \quad (6.18)$$

In this case: $\Xi[\Psi + \eta] \geq \Xi[\Psi]$ which, as η is an arbitrary function, implies that Ξ attains its minimum for the Ψ satisfying Eq. (6.18). The problem of finding the solution to Eq. (6.18) is thus converted to a minimum value problem.

If Eq. (6.3) is inserted in (6.16) the nodal stream function values are determined by

$$\frac{\partial}{\partial \psi_k} \Xi(\vec{\psi}) = 0 ; k = 1, \dots, m \quad (6.19)$$

As $\partial \Psi / \partial \psi_j = M_j$ Eq. (6.19) becomes

$$\begin{aligned} \sum_{j=1}^m \psi_j \iint_D \left(\frac{\partial M_j}{\partial r} \frac{\partial M_k}{\partial r} + \frac{\partial M_j}{\partial z} \frac{\partial M_k}{\partial z} \right) \frac{dz dr}{r} \\ + \sum_{j=1}^m g_j \iint_D M_j M_k dz dr + \sum_{j=1}^m \sigma_j \oint_C M_j M_k ds = 0 \end{aligned} \quad (6.20)$$

which is the same result as when the Galerkin Method was used, Eq. (6.14).

It may be noted that g is assumed not to be a function of Ψ here. Ξ is therefore not the true functional to the differential equation (6.1).

6.3. The Algorithms

When carrying out the calculations the quasiorthogonal net introduced in Fig. 3.4 is used but now the i-lines also remain stationary; they are not streamlines now as they were when the SCM was used. At the intersections of the k- and i-lines the nodes are located. Nodal quantities such as Γ and ζ_θ are obtainable by performing the differentiations in the (s, v) coordinate system, Eqs. (3.51), (3.56), and (3.52). The streamlines are traceable just as they were when the SCM was used.

The domain D is divided into small triangles as shown in Fig. 6.2. The stream function in a certain triangle is written analogously to Eq. (6.3) as

$$\Psi(r, z) = \sum_{i \in T} \psi_i N_i(r, z) \quad (6.21)$$

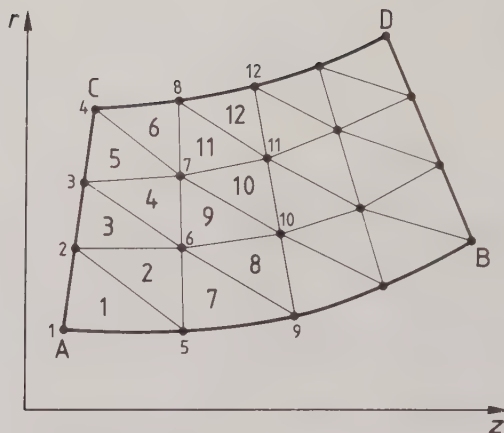


Fig. 6.2. The finite elements

where T is a set containing the three numbers of the involved nodes, for instance for the ninth triangle in Fig. 6.2, $T_9 = \{6, 10, 7\}$.

The N functions also fulfil Eq. (6.2), that is they are 1 in one node and 0 in the other two. If they are assumed to vary linearly with r and z

$$\left\{ \begin{array}{l} N_i(r, z) = a_i + b_i r + c_i z \\ a_i = \frac{r_{i-1} z_{i+1} - r_{i+1} z_{i-1}}{2A} \\ b_i = \frac{z_{i-1} - z_{i+1}}{2A} \\ c_i = \frac{r_{i+1} - r_{i-1}}{2A} \\ 2A = \sum_{i \in T} (r_{i-1} z_{i+1} - r_{i+1} z_{i-1}) \end{array} \right. \quad (6.22)$$

The subscripts $i+1$ and $i-1$ denote respectively nodes in front of and behind the i 'th node where the direction is counter-clockwise, that is the same as the curve C in Fig. 6.1. A is the area of the actual triangle.

Adding the Eq. (6.21) for all the triangles yields

$$\Psi(r, z) = \sum_{e=1}^n \sum_{i \in T_e} \psi_i N_i^{(e)}(r, z) \quad (6.23)$$

where superscript e denotes the number of the triangle. The total number of triangles is n . A certain $N^{(e)}$ function is only defined according to Eq. (6.22) in the e 'th triangle, elsewhere it is zero.

Comparing Eqs. (6.23) and (6.3) gives

$$M_k(r, z) = \sum_{e \in U_k} N_k^{(e)}(r, z) \quad (6.24)$$

The function M_k consists of those N functions which are 1 in the node k . The set U_k contains the numbers of these triangles, for instance $U_6 = \{2, 7, 8, 9, 4, 3\}$ in Fig. 6.2, and hence M_6 may be visualized as a pyramid with an irregular hexagon as base.

When evaluating Eq. (6.15) it is convenient to proceed triangle by triangle

$$\left\{ \begin{aligned} k_{kj}^{(e)} &= \iint_{D_e} \left(\frac{\partial N_k^{(e)}}{\partial r} \frac{\partial N_j^{(e)}}{\partial r} + \frac{\partial N_k^{(e)}}{\partial z} \frac{\partial N_j^{(e)}}{\partial z} \right) \frac{dz dr}{r} \\ h_{kj}^{(e)} &= \iint_{D_e} N_k^{(e)} N_j^{(e)} dz dr \\ d_{kj}^{(e)} &= \oint_{C_e} N_k^{(e)} N_j^{(e)} ds \quad ; \quad k, j \in T_e \end{aligned} \right. \quad (6.25)$$

A certain triangle contributes to nine of the matrix elements. Its contribution is added to those rows, k , and columns, j , which are determined by the three numbers in T_e .

Insertion of Eq. (6.22) in the first part of (6.25) yields

$$\left\{ \begin{aligned} k_{kj}^{(e)} &= (b_k^{(e)} b_j^{(e)} + c_k^{(e)} c_j^{(e)}) I_{1/r}^{(e)} \quad ; \quad k, j \in T_e \\ I_{1/r}^{(e)} &= \iint_{D_e} \frac{dz dr}{r} = \sum_{i \in T_e} \left\{ (z_{i+1} - z_i) + \frac{z_i r_{i+1} - z_{i+1} r_i}{r_{i+1} - r_i} \ln \frac{r_{i+1}}{r_i} \right\} \end{aligned} \right. \quad (6.26)$$

Evaluation of the second part of Eq. (6.25) gives

$$h_{kj}^{(e)} = \begin{cases} \frac{A_e}{6} & \text{for } k = j \\ \frac{A_e}{12} & \text{for } k \neq j \quad ; \quad k, j \in T_e \end{cases} \quad (6.27)$$

The third part of Eq. (6.25) is only defined for nodes on the boundary. The numbers of these nodes are contained in the set T_C . Between two nodes on the boundary the N functions are

$$N_k(s) = \frac{s - s_j}{s_k - s_j} \quad (6.28)$$

where s is a coordinate along the boundary. Insertion in Eq. (6.25) gives

$$\begin{cases} d_{kk}^{(e)} = \frac{|s_k - s_j|}{3} \\ d_{kj}^{(e)} = \frac{|s_k - s_j|}{6} \end{cases} ; k, j \in T_e \cap T_C \quad (6.29)$$

At the borderlines between the turbines two k-lines, as previously described in Sec. 5.1, are located at the same position and hence there are duplicate nodes here accounting for the flow respectively just before the entry shock, A, and just behind it, B. These duplicates are needed as $g_A \neq g_B$ in general but the stream function is continuous, $\Psi_A = \Psi_B$. This condition is imposed by treating the B nodes as A nodes when evaluating the $\mathbf{K}$ matrix.

If finally a set T_b , which contains the numbers of the nodes where ψ is known, $T_b \subset T_C$, is introduced the Eq. (6.15) converts to

$$\begin{aligned} \mathbf{K} \vec{\psi}_R &= \vec{x} \\ x_k &= -\sum_{j=1}^m h_{kj} g_j - \sum_{j=1}^m d_{kj} \Delta q_j - \sum_{j \in T_b} k_{kj} \psi_j ; k \notin T_b \end{aligned} \quad (6.30)$$

The original system of equations is thus reduced so that only the unknown stream function values, $\vec{\psi}_R$, are left.

The solution of Eq. (6.30) is divided into two steps

$$\begin{cases} \mathbf{K} = \mathbf{L} \mathbf{R} \\ \mathbf{L} \vec{y} = \vec{x} ; \mathbf{R} \vec{\psi}_R = \vec{y} \end{cases} \quad (6.31)$$

The $\mathbf{K}$ matrix is expressed as a product of two matrices, $\mathbf{L}$ and $\mathbf{R}$, which fulfil

$$\left\{ \begin{array}{l} k_{kj} = \sum_{p=1}^{j-1} l_{kp} r_{pj} + l_{kj} r_{jj} \text{ for } j \leq k \\ k_{kj} = \sum_{p=1}^{k-1} l_{kp} r_{pj} + r_{kj} \text{ for } j \geq k \\ l_{kj} = 0 \text{ for } j > k ; l_{kk} = 1 \\ r_{kj} = 0 \text{ for } j < k \end{array} \right. \quad (6.32)$$

that is, the $\mathbf{K}$ matrix is replaced by two triangular ones, allowing Eq. (6.31) to be written

$$\left\{ \begin{array}{l} y_k = x_k - \sum_{p=1}^{k-1} l_{kp} y_p ; k = 1, \dots, m ; k \notin T_b \\ \psi_{R,k} = \frac{1}{r_{kk}} (y_k - \sum_{p=k+1}^m r_{kp} \psi_{R,p}) ; k = m, \dots, 1 ; k \notin T_b \end{array} \right. \quad (6.33)$$

Eq. (6.32) has to be evaluated only once for a certain row of turbines, while Eq. (6.33) is evaluated whenever ψ_R is to be obtained.

The matrices involved usually have a band structure, that is, $k_{kj} = 0$ for $|k - j| > b$ where b is the band width. The band width can be made small by numbering the nodes in such a way that the difference in nodal numbers for neighbouring nodes is as small as possible. Only the parts of the matrices within the band have to be considered, thus reducing the size of the computer programs involved.

The stream function obtained, Ψ_R , suggests a change which is effectuated to an extent regulated by a damping coefficient Λ . A new stream function is thus

$$\Psi^{(n+1)} = \Psi^{(n)} + \Lambda (\Psi_R - \Psi^{(n)}) \quad (6.34)$$

6.4. Qualitative Analysis of the Convergence

The working of the FEM here employed can be studied qualitatively using the same assumptions as in Sec. 5.2, that is r and φ are treated as constants and λ is approximately zero.

The functional (6.16) may be expressed in (s, v) coordinates with the aid of Eqs. (3.49), (3.4), and (4.29).

$$\Xi[\Psi] = \iint_D \left(\frac{1}{2r} \Upsilon[\Psi, \Psi] + g \Psi \right) \cos \lambda_v ds dv \quad (6.35)$$

where the contour integral is omitted, as Ψ is assumed to be prescribed on the boundary henceforth.

Further assumptions are that $\omega = 0$, $\lambda_v \approx 0$, and that the influence from the v coordinate is minor, thus yielding

$$\begin{cases} \Xi[\Psi] \approx \iint_D \left(\frac{1}{2r} \left(\frac{\partial \Psi}{\partial s} \right)^2 + g \Psi \right) ds dv \\ g \approx - \frac{\partial^2 \Psi}{\partial s^2} \frac{\tan^2 \varphi_v}{r} + r \frac{dH}{d\Psi} \end{cases} \quad (6.36)$$

where also Eqs. (3.52), (3.51), (3.9), and (3.58) have been used.

Just as in Sec. 5.2 the actual stream function may be written as a sum of the correct function and a deviation. There are no deviations at the boundaries, $s = 0$ and $s = b$, see Fig. 5.3, for which reason the deviation may be written as a sine series and hence the actual stream function is

$$\Psi = \Psi_C + \sum_{j=1}^{\infty} e_j \sin \frac{j\pi s}{b} \quad (6.37)$$

Insertion in Eq. (6.36) gives

$$g \approx g_C + \sum_{j=1}^{\infty} \left(\frac{\tan^2 \varphi_v}{r} \left(\frac{j\pi}{b} \right)^2 e_j + r H_C'' e_j \right) \sin \frac{j\pi s}{b} \quad (6.38)$$

the quantity g is evaluated using the accessible stream function $\Psi^{(n)}$ and then a corrective stream function Ψ_R is calculated. This is determined by its Fourier coefficients $e_{R,k}$ which in their turn are determined by the condition $\partial \Xi / \partial e_{R,k} = 0$;

compare with Eq. (6.19). Insertion in Eq. (6.36) gives

$$\frac{\partial \Xi}{\partial e_{R,k}} = \iint_D \left(\frac{1}{r} \frac{\partial \Psi_R}{\partial s} \frac{\partial^2 \Psi_R}{\partial s \partial e_{R,k}} + g \frac{\partial \Psi_R}{\partial e_{R,k}} \right) ds dv = 0$$

and further insertion of Eqs. (6.37) and (6.38) and partial integration of the first term yields

$$\begin{aligned} & \left\{ \left[\frac{1}{r} \frac{\partial \Psi_C}{\partial s} \sin \frac{k \pi s}{b} \right]_0^b - \int_0^b \left(\frac{1}{r} \frac{\partial^2 \Psi_C}{\partial s^2} - g_C \right) \sin \frac{k \pi s}{b} ds \right. \\ & + \sum_{j=1}^{\infty} \left(\frac{e_{R,j}}{r} \frac{j k \pi^2}{b^2} \int_0^b \cos \frac{j \pi s}{b} \cos \frac{k \pi s}{b} ds \right) \\ & \left. + \sum_{j=1}^{\infty} \left(\frac{\tan^2 \varphi_v}{r} \left(\frac{j \pi}{b} \right)^2 + r H_C'' \right) e_j^{(n)} \int_0^b \sin \frac{j \pi s}{b} \sin \frac{k \pi s}{b} ds \right\} \int dv = 0 \end{aligned}$$

The second derivative H_C'' is assumed to be constant. As the correct stream function satisfies Eq. (6.1) exactly, the final result is

$$e_{R,k} = -e_k^{(n)} \left(\tan^2 \varphi_v + \left(\frac{b r}{k \pi} \right)^2 H_C'' \right) \quad (6.39)$$

The new stream function is obtained by Eq. (6.34) and hence its deviation has the Fourier coefficients

$$e_k^{(n+1)} = e_k^{(n)} \left\{ 1 - \Lambda \left(\frac{1}{\cos^2 \varphi_v} + \left(\frac{b r}{k \pi} \right)^2 H_C'' \right) \right\} \quad (6.40)$$

which should be compared with Eq. (5.30). The most significant difference is that no deterioration of the convergence is indicated for the FEM when the number of k-lines is increased.

Considering the first coefficient, the best damping coefficient is

$$\Lambda_o = \frac{1}{\frac{1}{\cos^2 \varphi_v} + \left(\frac{b r}{\pi} \right)^2 H_C''} \quad (6.41)$$

This indicates that the damping coefficient should be lower the bigger $|\varphi_v|$ is, which is opposite to the case for the SCM, see Eq. (5.31).

If the second derivative of the rothalpy is too low, the calculation method will fail; the limit for convergence with $\Lambda > 0$ is

$$H_C'' > - \left(\frac{\pi}{b r \cos \varphi_v} \right)^2 \quad (6.42)$$

The through flow area A is approximately $2 \pi r b$ which, inserted in the solvability criterion, gives

$$H_C'' > - \frac{4 \pi^4}{A^2 \cos^2 \varphi_v} \quad (6.43)$$

7. The Flow in a Torque Converter

7.1. The Circulating Through Flow

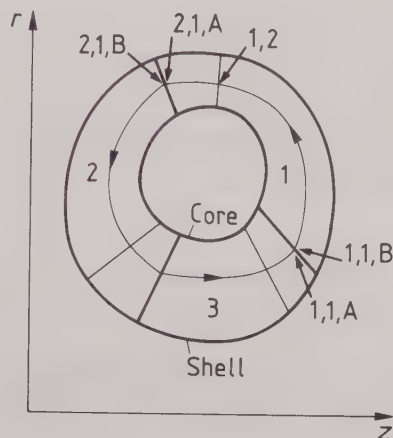


Fig. 7.1. Meridional cross section of a three-element torque converter

When a row of turbines are considered the through flow usually is indeterminate, that is Ψ is known or assigned at the walls AB and CD, see Fig. 3.1. Furthermore the specific energy I and the specific moment of momentum Γ may be assumed to be known in front of the row. None of these properties are known a priori in the case of a torque converter.

Fig. 7.1 shows the meridional cross section of a three-element torque converter. A streamline, which is a closed curve when the flow is stationary, is also shown. The points where the streamline enters and leaves the j 'th turbine are denoted by the subscripts $(j, 1)$ and $(j, 2)$. The subscripts A and B introduced previously denote a point just before and just after an entry shock. Thus the streamline is within the j 'th turbine when the subscript $(j, 1, B) \rightarrow (j, 2)$ and passes the duct between the j 'th and $j + 1$ 'th turbine when $(j, 2) \rightarrow (j + 1, 1, A)$.

If the dissipation is assumed to be small in the ducts between the turbines, Eqs. (3.45) and (3.42) give

$$\begin{cases} \Gamma_{j,1,A} = \Gamma_{j-1,2} \\ H_{j,1,A} = H_{j-1,2} \end{cases} \quad (7.1)$$

Within and at the entrance of a turbine, Eqs. (3.42), (4.25), and (4.34) give

$$H_{j,2} - H_{j,1,B} = - \int_{m_{j,1}}^{m_{j,2}} \Phi \frac{c_m}{\cos^2 \varphi_m} dm = -\Delta I_{D,j} \quad (7.2)$$

$$H_{j,1,B} - H_{j,1,A} = -\Delta I_{S,j,1} + (\omega_{j-1} - \omega_j) \Gamma_{j,1,A} \quad (7.3)$$

The torque converter has N turbines and as the N 'th turbine is followed by the first the identity $H_{N+1,2} = H_{1,2}$ may be expressed by adding the Eqs. (7.2), (7.3), and (7.1) successively yielding

$$-\sum_{j=1}^N \Delta I_{D,j} - \sum_{j=1}^N \Delta I_{S,j} + \sum_{j=1}^N (\omega_{j-1} - \omega_j) \Gamma_{j-1,2} = 0 \quad (7.4)$$

Eqs. (7.1), (7.2), and (7.4) apply along a certain streamline. Integrating the last of them with respect to Ψ and employing Eqs. (3.79), (3.90), and (3.80) gives

$$\sum_{j=1}^N (\Delta W_{D,j} + \Delta W_{S,j}) = \sum_{j=1}^N \omega_j (L_{j,2} - L_{j-1,2}) = \sum_{j=1}^N M_j \omega_j \quad (7.5)$$

which is the power balance for the entire machine.

Two adjacent streamlines may be characterized by their stream function values Ψ and $\Psi + \Delta\Psi$. The mass flow $\Delta\dot{m}$ which is locked up between them may be expressed with Eqs. (3.21) and (3.19)

$$\Delta\Psi = \frac{\Delta\dot{m}}{2\pi\rho} = r c_m \cos \lambda \Delta s = r_0 c_{m,0} \cos \lambda_0 \Delta s_0$$

where the subscript 0 refers to a certain k-line. Introducing the area coefficient a the meridional velocity along a certain streamline may be written

$$\begin{cases} c_m = \frac{c_{m,0}}{a} \\ a = \frac{r \cos \lambda}{r_0 \cos \lambda_0} \frac{ds}{ds_0} \end{cases} \quad (7.6)$$

Now the specific moment of momentum may be written

$$\begin{cases} \Gamma = r^2 \omega + \alpha c_{m,0} \\ \alpha = \frac{r \tan \varphi_m}{a} \end{cases} \quad (7.7)$$

and the specific energy loss due to the entry shock

$$\begin{cases} \Delta I_{S,j} = \frac{\kappa_j^*}{2} \left\{ \frac{1}{r_{j,1}} \left(r_{j-1,2}^2 \omega_{j-1} - r_{j,1}^2 \omega_j + c_{m,0} (\alpha_{j-1,2} - \alpha_{j,1}^*) \right) \right\}^2 \\ \alpha^* = \alpha_v + \frac{r \cos \lambda_v}{a} (\tan \lambda^* - \tan \lambda_v) \tan \varphi_s \end{cases} \quad (7.8)$$

where Eqs. (5.19), (7.1), (7.7), (4.31), (3.8), and (7.6) have been used.

Define finally

$$\Omega = 2 \sum_{j=1}^N \Delta I_{S,j} - 2 \sum_{j=1}^N (\Gamma_{j,2} - \Gamma_{j-1,2}) \omega_j \quad (7.9)$$

which, inserted in Eq. (7.4), gives

$$\Omega = -2 \sum_{j=1}^N \Delta I_{D,j} = -2 \Delta I_D \quad (7.10)$$

Insertion of Eqs. (7.7) and (7.8) in (7.9) yields the quadratic equation

$$\begin{cases} \Omega = A c_{m,0}^2 - 2 B c_{m,0} - C \\ A = \sum_{j=1}^N \kappa_j^* \left(\frac{\alpha_{j-1,2} - \alpha_{j,1}^*}{r_{j,1}} \right)^2 \\ B = \sum_{j=1}^N \omega_j B_j \\ B_j = \alpha_{j,2} - \alpha_{j-1,2} + \kappa_j^* (\alpha_{j-1,2} - \alpha_{j,1}^*) - \kappa_{j+1}^* \left(-\frac{r_{j,2}}{r_{j+1,1}} \right)^2 (\alpha_{j,2} - \alpha_{j+1,1}^*) \\ C = \sum_{j=1}^N \omega_j^2 C_{j,j} - \sum_{j=1}^N \omega_j \omega_{j+1} C_{j,j+1} \\ C_{j,j} = 2 r_{j,2}^2 - \kappa_j^* r_{j,1}^2 - \kappa_{j+1}^* \frac{r_{j,2}^4}{r_{j+1,1}^2}, \quad C_{j,j+1} = 2 r_{j,2}^2 (1 - \kappa_{j+1}^*) \end{cases} \quad (7.11)$$

Solving this with respect to $c_{m,0}$ and using Eq. (7.10) gives

$$c_{m,0} = \frac{B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 + \frac{C - 2 \Delta I_D}{A}} \quad (7.12)$$

In general the solution of this iterative, as ΔI_D is influenced by $c_{m,0}$, see Eq. (7.2).

A special case frequently used is to let a single central streamline represent the entire flow, thus reducing the problem to a one-dimensional one. In this case the variation of c_m with respect to s is not calculated.

In Refs. [4], [8], [19], [24], [26], and [27] this concept is employed. The energy loss ΔI_D is often calculated there by using the corresponding equations for friction losses in pipes. The pressure drop Δp along a pipe of length Δl is

$$\Delta p = \lambda_D \frac{\Delta l}{D_h} \rho \frac{w^2}{2} \quad (7.13)$$

where λ_D is a friction coefficient and D_h the hydraulic diameter. If the through flow area of the pipe is A the frictional force is $\Delta p A$ and distributing this over the contained mass of fluid gives

$$d = \frac{\Delta p A}{\rho \Delta l A}$$

Combining Eqs. (3.41) and (7.13) yields

$$\Phi = \frac{\lambda_D w}{2 D_h} \quad (7.14)$$

which inserted in Eq. (7.2) gives

$$\Delta I_{D,j} = \frac{c_{m,0}^2}{2} \int_{m_{j,1}}^{m_{j,2}} \frac{\lambda_D}{D_h a^2 \cos^3 \varphi_m} dm \quad (7.15)$$

The friction coefficient is often assumed to be independent of the flow velocity, allowing ΔI_D to be incorporated in $A c_{m,0}^2$, see Eqs. (7.10) and (7.11).

Eq. (7.12) sometimes proposes two values for $c_{m,0}$. If the non-stationary terms omitted from Eq. (7.2) are also included, that is returning to Eq. (3.40) and assuming the geometry of the streamlines to be independent of time, Eq. (7.10) becomes

$$\left\{ \begin{array}{l} \frac{\partial c_{m,0}}{\partial t} \sum_{j=1}^N S_j + \sum_{j=1}^N Y_j \frac{\partial \omega_j}{\partial t} + \Delta I_D + \frac{1}{2} \Omega = 0 \\ S_j = \int_{m_{j,1}}^{m_{j,2}} \frac{1 + \tan^2 \varphi_m}{a} dm \\ Y_j = \int_{m_{j,1}}^{m_{j,2}} r \tan \varphi_m dm \end{array} \right. \quad (7.16)$$

This equation does not include the ducts between the turbines but as these are usually small the error is not fatal.

Assume that the flow velocity is a sum of a time-independent value and a small time-dependent one

$$\left\{ \begin{array}{l} \tilde{c}_{m,0} = c_{m,0, \text{steady}} + \Delta c_{m,0} \\ \frac{\partial \tilde{c}_{m,0}}{\partial t} = \frac{\partial \Delta c_{m,0}}{\partial t} \end{array} \right.$$

If ΔI_D is assumed to be included in Ω and if all ω_j 's are independent of time, Eq. (7.16) becomes, as $\Omega_{\text{steady}} = 0$

$$\frac{\partial \Delta c_{m,0}}{\partial t} \sum_{j=1}^N S_j + \frac{1}{2} \left(\frac{\partial \Omega}{\partial c_{m,0}} \right)_{\text{steady}} \Delta c_{m,0} \approx 0 \quad (7.17)$$

If the steady flow were disturbed in some way, the disturbance would vanish if the coefficients in Eq. (7.17) have the same sign, which as all $S_j > 0$ implies

$$\left\{ \begin{array}{l} \left(\frac{\partial \Omega}{\partial c_{m,0}} \right)_{\text{steady}} > 0 \\ \Leftrightarrow \\ c_{m,0, \text{steady}} > \frac{B}{A} \end{array} \right. \quad (7.18)$$

Thus the upper root of (7.12) is always the stable one.

The stream function is now obtained by Eq. (3.19); at the 0-k-line is

$$\Psi(s_0) = \psi_{\text{shell}} + \int_{s_{0,\text{shell}}}^{s_0} c_{m,0} r_0 \cos \lambda_0 ds \quad (7.19)$$

The value ψ_{shell} can be chosen arbitrarily.

Evaluation of Eq. (6.30) presupposes nodal values of the stream function which are to be improved. The values obtained from the preceding iteration step do not quite fit as the stream function values connected with the traced streamlines, Eq. (7.19), have been changed since then. However, the positions of the streamlines are known from the preceding step and by means of the refreshed stream function values and the interpolation polynomial (5.6), new nodal values are obtained, see Fig. 7.2 function $\Psi(s)$. If however the streamlines were re-

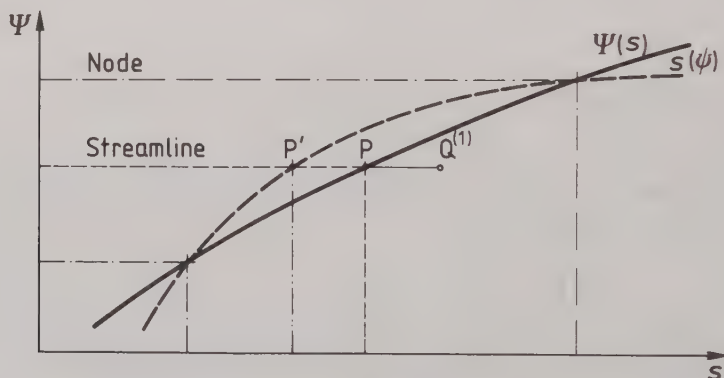


Fig. 7.2. Determination of nodal values, $\Psi(s)$, and streamline positions, $s(\psi)$

traced by inverse interpolation with the aid of these nodal values, function $s(\psi)$, which would be the case if the proposed nodal values turned out to be the correct ones, a false streamline shift would occur. Fig. 7.2 shows how a streamline at P is retraced to P'. The deviation is due to the numerical approximations but can be counteracted by performing the first interpolation with modified streamline positions s_Q , see Fig. 7.2. This is chosen according to

$$|Q^{(\mu+1)} Q^{(\mu)}| = |P P'|, \quad Q^{(0)} = P$$

giving the iteration formula

$$s_Q^{(\mu+1)} = s_Q^{(\mu)} + (s - s_{P'}) ; \quad s_Q^{(0)} = s \quad (7.20)$$

The procedure is repeated until P' and P are sufficiently close; this is usually a fast converging process.

When the streamlines are displaced their stream function values are affected and hence also the total through flow. Assume that the streamline positions are changed by a function $\delta(s_0)$, which has to be zero at the shell and the core as these are fixed. This produces changes of radius, area coefficient, and flow direction, angle λ . Neglecting the change of the quantity $\cos \lambda$ Eqs. (3.3) and (7.6) give

$$\begin{cases} \Delta r = \delta \sin \alpha \\ \Delta a = a \frac{\delta \sin \alpha}{r} + \frac{a}{\frac{ds}{ds_0}} \frac{d\delta}{ds_0} \end{cases} \quad (7.21)$$

Assume furthermore that $s(s_0) = b s_0/b_0$ and $\delta(s_0) = \hat{\delta} \sin(\pi s_0/b_0)$ giving $\|d\delta/ds_0\| = \pi \|\delta\|/b_0$ and hence

$$\left\| \frac{\Delta a}{a} \right\| \approx \left\| \frac{\delta \sin \alpha}{r} + \pi \frac{b_0}{b} \frac{\|\delta\|}{b_0} \right\| \approx \left\| \frac{\Delta r}{r} \right\| \left(1 + \frac{\pi r}{b \sin \alpha} \right) \quad (7.22)$$

As frequent values of r/b are $2 \div 7$ the change due to $d\delta/ds_0$ is the most significant. If $c_{m,0}$ is assumed to be a function of A , B , and C primarily the change due to δ would be

$$\Delta c_{m,0} = \sum_{j=1}^N \sum_{l=1}^2 \left(\frac{\partial c_{m,0}}{\partial A} \frac{\partial A}{\partial a_{j,l}} + \frac{\partial c_{m,0}}{\partial B} \frac{\partial B}{\partial a_{j,l}} \right) \frac{r_{j,l} \cos \lambda_{j,l}}{r_0 \cos \lambda_0} \frac{d\delta_{j,l}}{ds_0} \quad (7.23)$$

Integrating this and considering everything except δ as approximately constant shows that $\Delta c_{m,0}$ produces no net through flow. Thus ψ_{core} is not expected to vary much due to the streamline shifts.

7.2. Determination of the Rothalpy

The calculation of a new set of nodal stream function values is carried out as described in Chap. 6. The streamline positions are adjusted in accordance with this at all k-lines except the one which is chosen to have the 0 subscript introduced in the preceding section. This implies that the traced streamlines always pass certain fixed positions on that k-line.

When solving Eq. (6.30) only the values at the shell and the core are imposed as boundary conditions. Eq. (6.30) implies that a new stream function Ψ_R is obtained from an old one, $\Psi^{(n)}$ using the equation: $\zeta_{\theta, \Psi} [\Psi_R] = \zeta_{\theta, F} [\Psi^{(n)}]$. If $\Psi_R \neq \Psi^{(n)}$ a modification is needed, which is done at the 0-k-line by changing the rothalpy there. Eq. (3.59) shows that the difference $\Delta \zeta_{\theta, F}$ corresponds to $-r d \Delta H / d\Psi$ and as the difference needed is $\zeta_{\theta, \Psi} [\Psi^{(n)}] - \zeta_{\theta, \Psi} [\Psi_R]$ the proper modification of the rothalpy is

$$\Delta H(\Psi) = \int_{\psi_{\text{shell}}}^{\Psi} \frac{1}{r} \zeta_{\theta, \Psi} [\Psi_R - \Psi^{(n)}] d\Psi \quad (7.24)$$

The new rothalpy is

$$H^{(n+1)} = H^{(n)} + \Lambda_H \Delta H \quad (7.25)$$

where a damping coefficient is also introduced.

This equation is performed only at the 0-k-line. The rothalpy is then calculated along the streamlines using Eqs. (3.42) and (4.25) and when a round is completed the original value is obtained again, as the circulating flow is determined from this condition.

As only derivatives and differences of the rothalpy are present a constant could be added to all the values without changing anything. Eq. (3.39) shows that this corresponds to different pressure levels in the fluid. Avoidance of cavitation phenomena however requires $p > 0$ everywhere and thus a certain pressure level is minimum, which is usually exerted by an auxiliary pump.

Eq. (6.34) is performed for all k-lines and the tracing of the streamlines is done with these values where the streamlines are characterized by the now relevant ones at the 0-k-line.

7.3. Qualitative Analysis of the Convergence

The analysis of the convergence done in Sec. 6.4 can be extended to include the changes of the rothalpy too. As $\psi_{\text{core}} \approx \psi_{\text{C,core}}$ Eq. (6.37) still applies. The rothalpy may analogously be written as a sum of the correct function and a deviation

$$H = H_C + \sum_{j=1}^{\infty} h_j \left(1 - \cos \frac{j \pi s}{b} \right) \quad (7.26)$$

The value at the shell, $s = 0$, is assumed to be a boundary condition which is unchanged throughout the calculation.

Eq. (6.38), with the use of (6.37), is replaced by

$$g \approx g_C + \sum_{j=1}^{\infty} \left\{ \left(\frac{\tan^2 \varphi_v}{r} \left(\frac{j \pi}{b} \right)^2 + r H_C'' \right) e_j + \frac{j \pi}{c_m b} h_j \right\} \sin \frac{j \pi s}{b} \quad (7.27)$$

which gives the Fourier coefficients for the renewal flow, compare Eq. (6.39)

$$e_{R,k} \approx -e_k^{(n)} \left(\tan^2 \varphi_v + \left(\frac{r b}{k \pi} \right)^2 H_C'' \right) - h_k^{(n)} \frac{r b}{k \pi c_m} \quad (7.28)$$

The renewal coefficients for the rothalpy are obtained using Eqs. (7.24), (3.58), (3.64), (3.49), and (3.52)

$$\frac{d \Delta H_R}{d \Psi} \approx \frac{1}{r c_m} \frac{\partial \Delta H_R}{\partial s} \approx -\frac{1}{r^2} \frac{\partial^2 (\Psi_R - \Psi^{(n)})}{\partial s^2}$$

Insertion of Eqs. (7.28) and (6.37) yields a solution of the type (7.26) with the coefficients

$$h_{R,k} = -\frac{c_m k \pi}{r b} \left(\frac{1}{\cos^2 \varphi_v} + \left(\frac{r b}{k \pi} \right)^2 H_C'' \right) e_k^{(n)} - h_k^{(n)} \quad (7.29)$$

and hence the new deviations are obtained from Eqs. (6.34) and (7.25)

$$\begin{cases} e_k^{(n+1)} = e_k^{(n)} \left\{ 1 - \Lambda \left(\frac{1}{\cos^2 \varphi_v} + \left(\frac{r b}{k \pi} \right)^2 H_C'' \right) \right\} - h_k^{(n)} \Lambda \frac{r b}{k \pi c_m} \\ h_k^{(n+1)} = h_k^{(n)} (1 - \Lambda_H) - e_k^{(n)} \Lambda_H \frac{k \pi c_m}{r b} \left(\frac{1}{\cos^2 \varphi_v} + \left(\frac{r b}{k \pi} \right)^2 H_C'' \right) \end{cases} \quad (7.30)$$

The last of these equations is used at the 0-k-line only, while the first applies at the others.

These equations can be assumed to correspond to the differential equations

$$\begin{cases} \frac{de_k}{dn} = -\Lambda \left(\frac{1}{\cos^2 \varphi_v} + \left(\frac{rb}{k\pi} \right)^2 H_C'' \right) e_k - \Lambda \frac{rb}{k\pi c_m} h_k \\ \frac{dh_{k,0}}{dn} = -\Lambda_H h_{k,0} - \Lambda_H \left(\frac{1}{\cos^2 \varphi_{v,0}} + \left(\frac{r_0 b_0}{k\pi} \right)^2 H_{C,0}'' \right) \frac{k\pi c_{m,0}}{r_0 b_0} e_{k,0} \end{cases} \quad (7.31)$$

If the correct flow were used at the 0-k-line, $e_{k,0} = 0$, their solutions would be

$$\begin{cases} h_{k,0}^{(n)} = h_{k,0}^{(0)} \exp(-\Lambda_H n) \\ e_k^{(n)} = e_k^{(0)} \exp(-\xi_k \Lambda n) + h_{k,0}^{(0)} \frac{\Lambda r b}{k\pi c_m} \frac{\exp(-\xi_k \Lambda n) - \exp(-\Lambda_H n)}{\xi_k \Lambda - \Lambda_H} \\ \xi_k = \frac{1}{\cos^2 \varphi_v} + \left(\frac{rb}{k\pi} \right)^2 H_C'' \end{cases} \quad (7.32)$$

This example shows that $\Lambda_H / \Lambda = 1 \div 2$ may be a suitable rule of thumb for selection of damping coefficients.

7.4. Determination of Angular Velocities

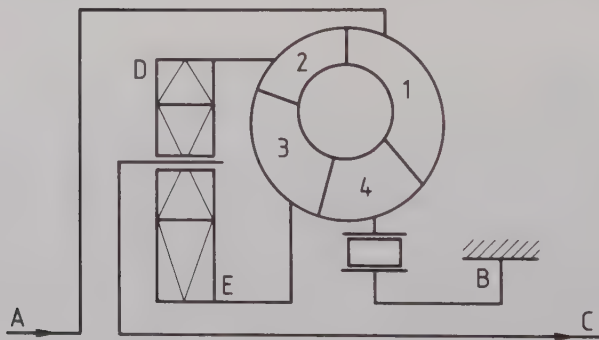


Fig. 7.3. A transmission consisting of a four-element torque converter, an epicyclic gear, and a unidirectional clutch

Usually the angular velocities of the turbines in the torque converters are known or assigned at the initiations of the calculation. Sometimes however the torques are imposed, instead of the angular velocities, thus indirectly governing these. Fig. 7.3 shows an example where the 2nd and the 3rd turbines are connected to the driven axle by an epicyclic gear and the 4th turbine to the casing by a unidirectional clutch.

The governing equations for the epicyclic gear are

$$\begin{cases} \frac{\omega_E - \omega_C}{\omega_D - \omega_C} = -\frac{Z_D}{Z_E} \\ M_E + M_D + M_C = 0 \\ E_E + E_D + E_C = 0 \end{cases} \quad (7.33)$$

which is the gear ratio relative to the planet carrier, Z denotes number of teeth for the actual gearwheel, and the torque and power balances in the absence of frictional effects. The last of these equations can be evolved to

$$\begin{aligned} (M_E + M_D + M_C) \omega_C + M_E (\omega_E - \omega_C) + M_D (\omega_D - \omega_C) &= \\ &= -M_E Z_D/Z_E + M_D = 0 \end{aligned}$$

Neglecting drag torques, the unidirectional clutch fulfils one of the conditions

$$\begin{cases} \nu M_4 \geq 0 \text{ and } \omega_4 = \omega_B \\ M_4 = 0 \text{ and } \nu(\omega_4 - \omega_B) > 0 \end{cases} \quad (7.34)$$

The first condition applies when the clutch is locked and the second when it overruns. The quantity ν is either +1 or -1 referring to the two possible lock directions of such a clutch. In this example $\nu = 1$ would be the most adequate choice.

Generally the equations determining the angular velocities may be written

$$\begin{cases} \sum_{j=1}^N d_{kj} \omega_j = \beta_k & k = 1, \dots, N_\omega \\ \sum_{j=1}^N d_{kj} M_j = 0 & k = N_\omega + 1, \dots, N \end{cases} \quad (7.35)$$

For the transmission in Fig. 7.3, if $\omega_B = 0$ and $M_4 \geq 0$,

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Z_D/Z_E & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -Z_D/Z_E & 0 \end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix} \omega_A \\ (1 + Z_D/Z_E) \omega_C \\ 0 \\ 0 \end{pmatrix}, \quad N_\omega = 3 \quad (7.36)$$

The condition $M_4 = 0$ would be imposed by using $N_\omega = 2$ above instead.

A mean value for the specific moment of momentums may be defined as

$$\overline{\Gamma} = \frac{L}{\dot{m}} = \frac{1}{\psi_{\text{core}} - \psi_{\text{shell}}} \int_{\psi_{\text{shell}}}^{\psi_{\text{core}}} \Gamma d\psi \quad (7.37)$$

see Eqs. (3.79) and (3.21). Now the torques for the turbines, see Eq. (3.80), are

$$M_j = \dot{m} (\overline{\Gamma}_{j,2} - \overline{\Gamma}_{j-1,2}) \quad (7.38)$$

Insertion of Eq. (3.34) in (7.37) and comparison with (7.7) leads to the definitions

$$\left\{ \begin{aligned} \overline{r^2} &= \frac{1}{\psi_{\text{core}} - \psi_{\text{shell}}} \int_{\psi_{\text{shell}}}^{\psi_{\text{core}}} r^2 d\psi \\ \overline{\alpha c_{m,0}} &= \frac{1}{\psi_{\text{core}} - \psi_{\text{shell}}} \int_{\psi_{\text{shell}}}^{\psi_{\text{core}}} \frac{r c_{m,0} \tan \varphi_m}{a} d\psi \\ \overline{\Gamma} &= \overline{r^2} \omega + \overline{\alpha c_{m,0}} \end{aligned} \right. \quad (7.39)$$

The mean value for the meridional flow velocity is defined analogously as

$$\overline{c_m} = \frac{1}{\psi_{\text{core}} - \psi_{\text{shell}}} \int_{\psi_{\text{shell}}}^{\psi_{\text{core}}} c_m d\psi \quad (7.40)$$

and finally

$$\left\{ \begin{array}{l} \bar{\alpha} = \frac{\overline{\alpha c_{m,0}}}{\overline{c_{m,0}}} \\ \bar{a} = \frac{\overline{c_{m,0}}}{\overline{c_m}} \\ \bar{r} = \sqrt{\overline{r^2}} \\ \overline{\tan \varphi_m} = \frac{\bar{\alpha} \bar{a}}{\bar{r}} \end{array} \right. \quad (7.41)$$

may be introduced. The mean shock factor is defined according to Eqs. (7.8) and (3.90) as

$$\bar{\kappa}_j = \frac{2 \Delta W_{s,j}}{\dot{m} \left(\frac{\bar{\Gamma}_{j-1,2} - \bar{\Gamma}_{j,1}}{\bar{r}_{j,1}} \right)^2} \quad (7.42)$$

Use of these mean values allows the circulating flow to be calculated as if it was concentrated to a single mean streamline. However, as $\bar{\Gamma}_{j-1,2} = \bar{\Gamma}_{j,1}$ does not implicate $\Gamma_{j-1,2} \equiv \Gamma_{j,1}$ and hence not $\Delta W_{s,j} = 0$ either, the values of $\bar{\kappa}$ will probably differ a good deal from those according to Eq. (5.19).

An effective through flow area at the 0-k-line is calculated from Eq. (3.22)

$$\left\{ \begin{array}{l} A_{0,\text{eff}} = \frac{\dot{V}}{\overline{c_{m,0}}} = \frac{2\pi (\psi_{\text{core}} - \psi_{\text{shell}})}{\overline{c_{m,0}}} \\ a_0 = \frac{A_{0,\text{eff}}}{A_0} \end{array} \right. \quad (7.43)$$

An area coefficient a_0 which is the quotient between the effective and the actual area is also introduced.

The quadratic equation (7.11) may also be written employing matrix notation, Ref. [27]

$$\left\{ \begin{array}{l} \vec{C}^T \mathbf{Q} \vec{C} = \overline{\Omega} \\ \vec{C} = (\overline{c_{m,0}}, \omega_1, \dots, \omega_N)^T \\ q_{0,0} = \overline{A} \\ q_{0,j} = q_{j,0} = -\overline{B_j} \quad j = 1, \dots, N \\ q_{jj} = -\overline{C_{j,j}} \quad j = 1, \dots, N \\ q_{j,j+1} = q_{j+1,j} = -\frac{1}{2} \overline{C_{j,j+1}} \quad j = 1, \dots, N \end{array} \right. \quad (7.44)$$

Insertion of Eq. (7.38) in (7.35) yields

$$\left\{ \begin{array}{l} \mathbf{G} \vec{C} = \vec{B} \\ \vec{B} = (\overline{c_{m,0}}, \beta_1, \dots, \beta_N)^T ; \beta_j = 0 \text{ for } j > N_\omega \\ g_{0,0} = 1 \\ g_{k,j} = d_{k,j} \quad k = 1, \dots, N_\omega ; j = 1, \dots, N \\ g_{k,0} = \sum_{j=1}^N d_{k,j} (\overline{\alpha_{j,2}} - \overline{\alpha_{j-1,2}}) \quad k = N_\omega + 1, \dots, N \\ g_{k,j} = (d_{k,j} - d_{k,j+1}) \overline{r_{j,2}^2} \quad j = 1, \dots, N ; k = N_\omega + 1, \dots, N \end{array} \right. \quad (7.45)$$

Those matrix elements not assigned explicitly are zero and as the N 'th turbine is followed by the first the element $g_{k,N+1}$ is $g_{k,1}$ etc. The matrix elements can be evaluated either using the mean values from Eqs. (7.39) - (7.42) or estimations on the basis of the geometrical properties of the torque converter. Combination of Eqs. (7.45) and (7.44) yields

$$\left\{ \begin{array}{l} \vec{C} = \mathbf{G}^{-1} \vec{B} \\ \vec{B}^T [(\mathbf{G}^{-1})^T \mathbf{Q} (\mathbf{G}^{-1})] \vec{B} = \overline{\Omega} \end{array} \right. \quad (7.46)$$

Thus the entire transmission including torque converter, gears etc. can be transformed into a fictive torque converter. The circulating flow is solved from the last of the Eqs. (7.46) and then the actual angular velocities from the first.

This calculation can be done before initiation of the main solution procedure and hence the angular velocities may be assumed to need only minor adjust-

ments during this procedure. Insertions of the intermediate values in Eq. (7.35) or (7.45) may yield a slightly erroneous result $\vec{B}^{(n)}$, which may be corrected by changing the velocities according to

$$\mathbf{G} \Delta \vec{C} = \mathbf{G} (\vec{C}_R - \vec{C}^{(n)}) = 0 - \vec{B}^{(n)}$$

If only the angular velocities are changed here this equation reduces to

$$\left\{ \begin{array}{l} \mathbf{T} \Delta \vec{\omega} = \mathbf{T} (\vec{\omega}_R - \vec{\omega}^{(n)}) = \Delta \vec{\beta} \\ t_{k,j} = g_{k,j} \quad k, j = 1, \dots, N \\ \Delta \beta_k = 0 \quad k = 1, \dots, N_\omega \\ \Delta \beta_k = -\frac{1}{\dot{m}} \sum_{j=1}^N d_{k,j} M_j^{(n)} \quad k = N_\omega + 1, \dots, N \end{array} \right. \quad (7.47)$$

The absolute values of $\vec{\beta}$, for instance according to Eq. (7.36), are not needed but the initial values of $\vec{\omega}$ have to be consistent with them. The new angular velocities are obtained from

$$\vec{\omega}^{(n+1)} = \vec{\omega}^{(n)} + \Lambda_\omega (\mathbf{T}^{-1} \Delta \vec{\beta}) \quad (7.48)$$

where a damping coefficient is also introduced.

Finally, a specific set of N_ω and d values implicates certain states of the uni-directional clutches involved. Therefore the satisfaction of the conditions also implicated (7.34) must be checked during the solution procedure.

At last, as all parts needed for the analysis of the flow in a torque converter have been worked out, the complete computing procedure can be constructed. Its outlines are shown in Fig. 7.4.

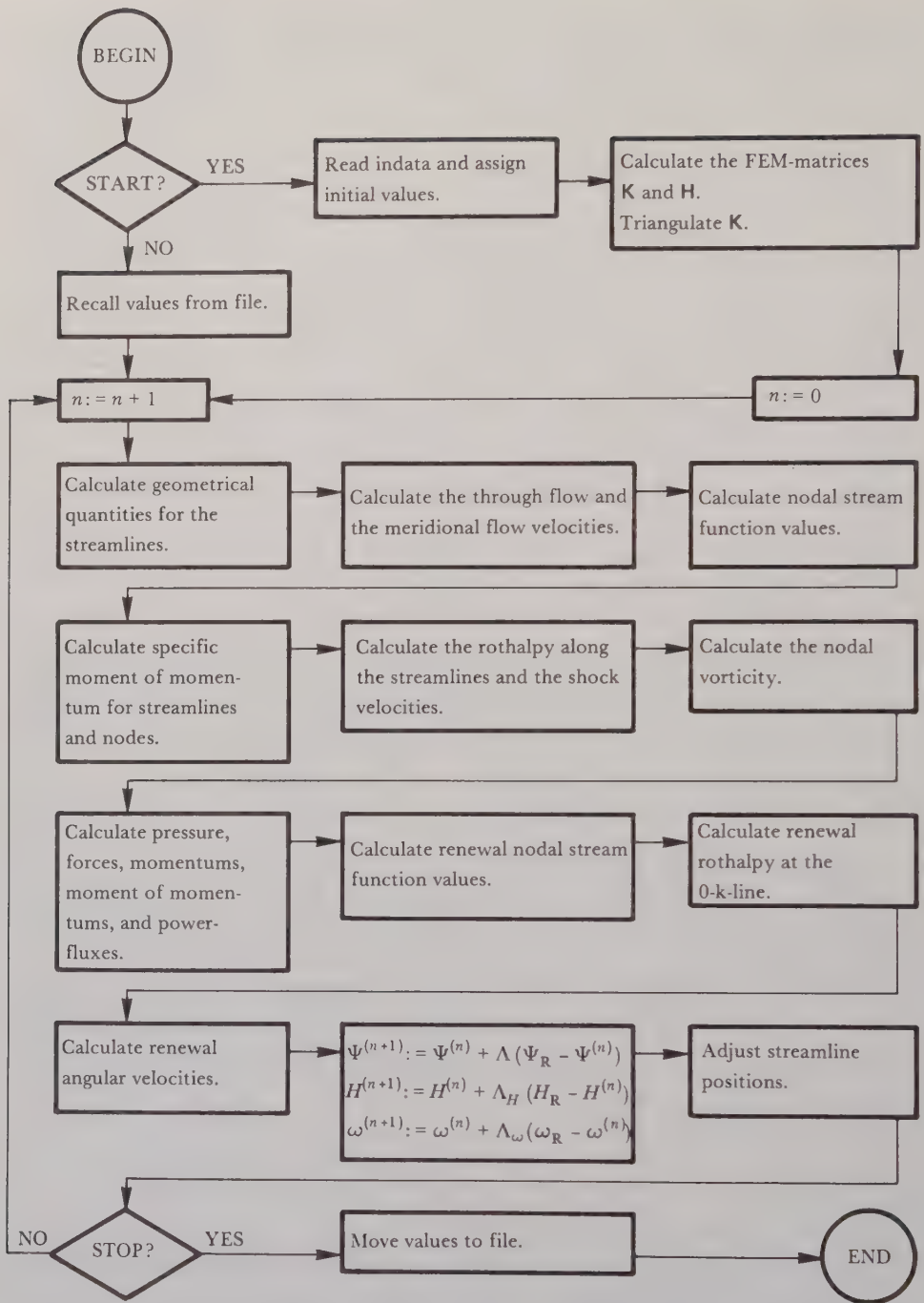


Fig. 7.4. Flow chart for the computing procedure

7.5 A Note on Non-Stationary Flow

An analysis of the non-stationary performance of a torque converter may be carried out by employing the concept of one-dimensional flow. Eq. (7.16) is then applied to a significative central streamline. The torques may be expressed by combining Eqs. (3.78), (7.7), and (7.38). The volume integral is evaluated as: $\iiint (\) dV = \int (\) A dm = A_0 \int (\) a dm$, where m is a length coordinate along the central streamline, A the through flow area according to $\dot{m} = \rho c_m A$, and a an area coefficient, $a = A/A_0 = c_{m,0}/c_m$, Eq. (7.6). The performance is thus governed by the $N + 1$ differential equations

$$\left\{ \begin{array}{l} \frac{\partial c_{m,0}}{\partial t} \sum_{j=1}^N \int_{m_{j,1}}^{m_{j,2}} \frac{dm}{a \cos^2 \varphi_m} + \sum_{j=1}^N \frac{\partial \omega_j}{\partial t} \int_{m_{j,1}}^{m_{j,2}} r \tan \varphi_m dm + \Delta I_D + \frac{1}{2} \Omega = 0 \\ M_j = \rho A_0 \left\{ \frac{\partial c_{m,0}}{\partial t} \int_{m_{j,1}}^{m_{j,2}} r \tan \varphi_m dm + \frac{\partial \omega_j}{\partial t} \int_{m_{j,1}}^{m_{j,2}} r^2 a dm \right\} + \dot{m} (\Gamma_{j,2} - \Gamma_{j-1,2}) \end{array} \right. \quad (7.49)$$

8. Test of the Calculation Methods

8.1. Test Example I

A simple test example was constructed to get an idea of the efficacy of the calculation procedure. The meridional geometry and flow velocities are determined by the equations

$$\left\{ \begin{array}{l}
 \Psi = -\frac{1}{2} B r^2 z \\
 c_r = \frac{1}{2} B r ; c_z = -B z ; \xi_\theta = 0 \\
 r_\psi = \sqrt{\frac{2\psi}{-Bz}} ; r_\alpha = a - (a+z) \cot(\alpha - \pi/2) \\
 r_{\psi,\alpha}^3 \tan(\alpha - \pi/2) + a r_{\psi,\alpha}^2 [1 - \tan(\alpha - \pi/2)] - 2\psi/B = 0 \\
 \lambda = \arctan(-r/2z) - \alpha + \pi/2 \\
 K = (3r/4z^2)(1 + r^2/4z^2)^{-3/2} \\
 s = (r - r_{AB})/\sin \alpha \\
 m = \int_{-1}^z \sqrt{1 + \psi/2B(-z)^3} dz \\
 r = (1 - \sigma) r_{AB} + \sigma r_{CD} ; \sigma = s/s_{CD} \\
 z = (1 - \sigma) z_{AB} + \sigma z_{CD} \\
 \beta = \arctan \left(\frac{dr}{d\alpha} / \frac{dz}{d\alpha} \right) \\
 v = \int_{\pi/2}^{\alpha} \sqrt{\left(\frac{dz}{d\alpha} \right)^2 + \left(\frac{dr}{d\alpha} \right)^2} d\alpha
 \end{array} \right. \quad (8.1)$$

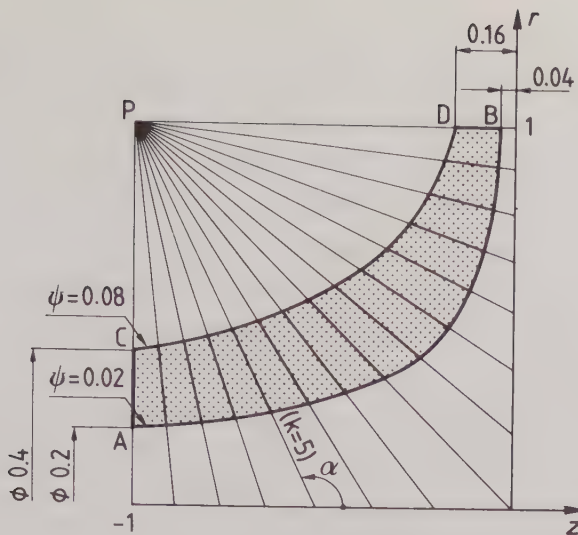


Fig. 8.1. Test turbine I

These equations generate the test turbine shown in Fig. 8.1 when $a = 1$, $B = 1$, $\psi_{AB} = 0.02$, and $\psi_{CD} = 0.08$.

The net introduced in Fig. 3.4 is generated by dividing the angle α , $\alpha \in [\pi/2, \pi]$, and the width along the k -lines, that is the width factor σ , $\sigma \in [0, 1]$, in a number of parts.

Now the blade surface angle $\Theta = C r^k (-z)^n$ is proposed. The specific moment of momentum and the blade forces can be calculated. Insertion in Eq. (3.70) shows that k and n could not be chosen arbitrarily; the chosen possibility is $n = (k + 1)/2$, furthermore $\omega = 0$ is also chosen. The rothalpy can now be evaluated yielding the complete result

$$\Theta = C r^k (-z)^{(k+1)/2} = C (2 \Psi/B)^{(k+1)/2} / r$$

$$\tan \varphi_r = C k r^k (-z)^{(k+1)/2} = C k (2 \Psi/B)^{(k+1)/2} / r$$

$$\tan \varphi_z = -C (k+1) r^{k+1} (-z)^{(k-1)/2} / 2 = -C (k+1) (2 \Psi/B)^{(k-1)/2} r^2 / 2$$

$$k_\theta = -B^2 C r^{k+1} (-z)^{(k+1)/2} / 4 = -B^2 C (2 \Psi/B)^{(k+1)/2} / 4$$

$$\Gamma = -B C r^{k+2} (-z)^{(k+1)/2} / 2 = -B C (2 \Psi/B)^{(k+1)/2} r / 2$$

$$H = B^2 C^2 r^2 (-z)^{k+1} / 4 = B^2 C^2 (2 \Psi/B)^{k+1} / 4 \quad (8.2)$$

The test is carried out by comparing the exact values with the approximative ones obtained from the numerical procedures. Table 8.1 shows a sample from the comparison of the meridional flow velocity c_m at the wall AB which is chosen because the deviations are mostly largest there. The prime test case has 10 i-lines and 15 k-lines, that is the net has 9×14 quasiorthogonals, see also Fig. 8.1. No blades exist in this case, $C = 0$. The number of traced streamlines is equal to the number of i-lines and their characteristic stream function values are those of the nodes on the first k-line.

In the second case the mesh density is doubled to 18×28 while in the third blades according to $k = -1.25$ and $C = -0.427994$ are considered. (The value of C emanates from the condition: $\tan \varphi_{r,\max} = 4$).

These three examples are all carried out twice, using the SCM and the FEM respectively. The initial streamline positions are chosen to be the i-lines and hence the positions are correct at the first k-line from the beginning.

		$c_{m,\text{approx}} - c_{m,\text{exact}}$					
		$\Theta = 0$				$\Theta = -.427994 r^{-5/4} (-z)^{-1/8}$	
		1. 9×14		2. 18×28		3. 9×14	
k	$c_{m,\text{exact}}$	SCM	FEM	SCM	FEM	SCM	FEM
1	1.004 988	—	—	—	—	—	—
2	.916 943	-.000 869	-.000 506	-.000 058	-.000 138	-.001 542	-.000 613
3	.829 468	-.000 517	-.000 756	-.000 128	-.000 204	.000 124	-.000 776
4	.741 164	-.001 011	-.000 780	-.000 236	-.000 214	-.000 046	-.000 878
5	.651 054	-.001 559	-.000 542	-.000 400	-.000 154	.000 486	-.000 709
6	.559 050	-.002 288	-.000 094	-.000 606	-.000 033	.000 914	-.000 245
7	.467 194	-.002 569	.000 263	-.000 727	.000 063	.001 877	.000 284
8	.382 362	-.001 159	.000 061	-.000 353	.000 005	.003 214	-.000 107
9	.319 530	.002 766	-.000 628	.000 834	-.000 178	.003 245	-.001 782
10	.296 202	.005 010	-.000 743	.001 457	-.000 195	.000 378	-.002 350
11	.312 338	.002 691	-.000 221	.000 685	-.000 053	-.001 913	-.001 153
12	.350 969	.000 136	-.000 003	.000 012	.000 001	-.002 348	-.000 194
13	.398 538	-.000 795	-.000 078	-.000 208	-.000 022	-.001 660	.000 440
14	.449 298	-.002 354	-.000 151	-.000 403	-.000 040	-.000 150	.001 247
15	.501 597	-.005 509	.000 006	-.002 670	.000 001	.003 548	.000 006

Table 8.1. Numerical errors in the meridional flow velocity at the wall AB for some test examples

Some general features are observable in Table 8.1. The error is reduced to about a fourth when the mesh density is doubled for both the methods. The FEM usually gives smaller errors than the SCM, but the comparison is not fair at the outlet as the correct boundary conditions are not imposed here for the SCM. The differences between the first and the third test cases are caused by the errors in the additional numerical derivations of the blade angles and the rothalpy. The magnitude of the errors is not changed significantly but the differences between the methods are reduced.

These comparisons favour the FEM only a little, but this method was chosen because its convergence qualities do not suffer when the mesh density is increased.

Fig. 8.2 shows the evolution of the maximum and the mean value of the quantity $|\psi_R - \psi^{(n)}|$ respectively. These values decline mainly as a geometrical series but the rate of decline decreases when the values vanish.

Finally, the damping coefficient Λ could have been increased more and the last five iterations in the referred case could have been omitted as they only present insignificant changes.

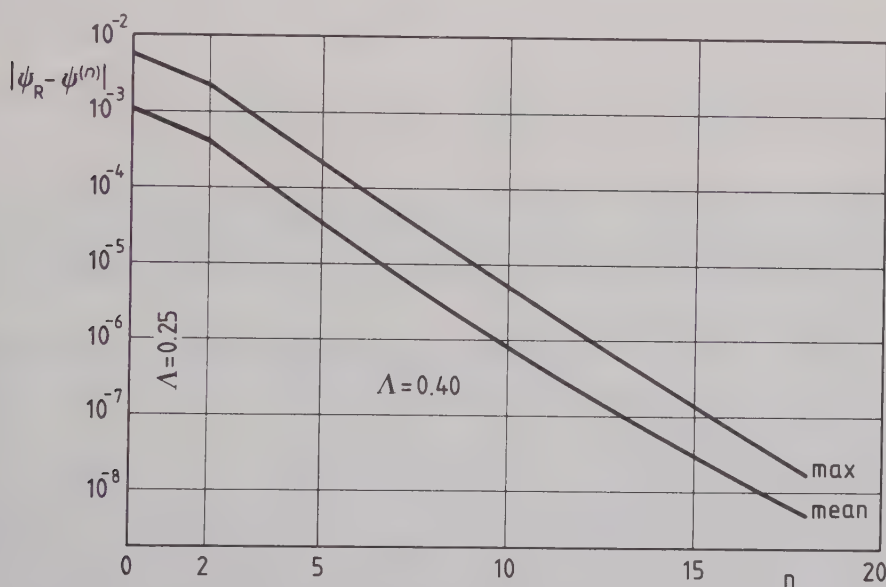


Fig. 8.2. The convergence of the FEM for the third test case, Table 8.1

8.2. Test Example II

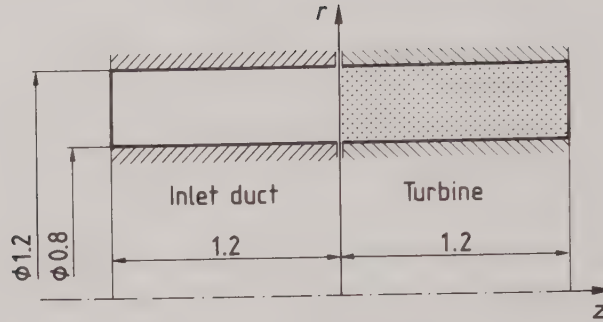


Fig. 8.3. Test turbine II

Fig. 8.3 shows a test example which was constructed in order to study the flow in the inlet region of a turbine. An axial flow turbine preceded by an annular duct was chosen to make comparisons with the example in Sec. 4.3 possible. This turbine has $\varphi_r = \text{const.}$, $\varphi_z = 0$, $\Gamma_{\text{inlet}} = \text{const.}$, and $c_{m,\text{inlet}} = 1$ and it is also assumed that the radius is almost constant, $r \gg b$, though this is not quite fulfilled in the example in Fig. 8.3.

The desired blade angles here are generated by the function: $\Theta = \tan \varphi_r \ln(r/0.8)$. At the inlet $\Gamma = r$ is chosen and hence $c_\theta = 1$, which corresponds to $\Gamma_A = 1$ in the example in Sec. 4.3. The flow velocities $c_r = 0$ and $c_z = 1$ correspond to the stream function: $\Psi = (r^2 - 0.8^2)/2$. This stream function inserted in Eq. (3.64) gives $\zeta_\theta = 0$ and now the rothalpy is obtainable from Eq. (3.60) and, as $\omega = 0$, is written: $H = 1 + \ln(r/0.8)$.

In the duct in front of the turbine the moment of momentum and rothalpy are constant along the streamlines and may be written

$$\begin{cases} \Gamma(\Psi) = \sqrt{2\Psi + 0.64} \\ H(\Psi) = 1 + \ln\sqrt{1 + \Psi/0.32} \end{cases} \quad (8.3)$$

The flow was calculated for four values of the angle φ_r , namely: 0, 15, 30, and 45°. Fig. 8.4 shows the streamlines for $\varphi_r = 0$ and 45°. It also shows the i- and k-lines used.

In Fig. 8.5 values obtained from the numerical calculations and from the example in Sec. 4.3 at the point: $z = 0$ and $r = 1$, that is in the middle of the inlet of the turbine, are compared. The list (4.53) is supplemented with the vorti-

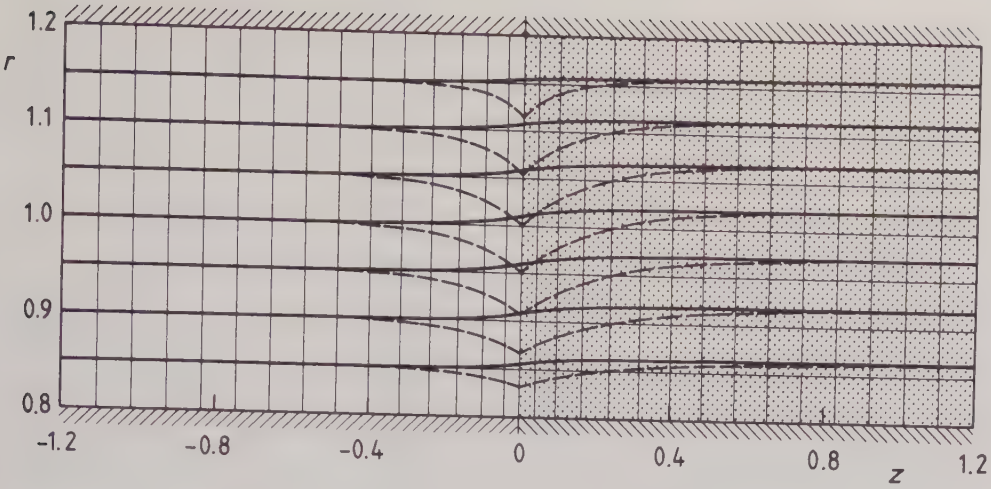


Fig. 8.4. Streamlines for the cases $\varphi_r = 0$ and 45° , solid and dashed lines respectively, and the i- and k-lines

city $\xi_{\theta B}$ and the curvatures K_A and K_B . The vorticity is evaluated by inserting Eqs. (4.46) in (3.30) and the curvatures by combining (3.24), (3.6), (3.4), and (3.1) giving

$$K = \frac{1}{c_m^3} \left\{ c_r \left(c_z \frac{\partial c_r}{\partial r} - c_r \frac{\partial c_z}{\partial r} \right) + c_z \left(c_z \frac{\partial c_r}{\partial z} - c_r \frac{\partial c_z}{\partial z} \right) \right\} \quad (8.4)$$

in which (4.41) and (4.45) are inserted. The actual figures to be inserted are $b = 0.4$, $s = 0.2$, $r = 1$, $z = 0$, $\Gamma_A = 1$, $\omega_B = 0$, and $c_\infty = 1$ and the evaluation also calls for the series

$$\left\{ \begin{aligned} \sum_{k=1}^{\infty} \frac{b \iota_k}{k \pi} \sin \frac{k \pi}{2} &= \frac{4b}{\pi^2} \quad 0.915966 \\ \sum_{k=1}^{\infty} \frac{k \pi \iota_k}{b} \sin \frac{k \pi}{2} \exp \left(-\frac{k \pi \cos \varphi_r z}{b} \right) &= \frac{2}{b \cosh \left(-\frac{\pi \cos \varphi_r z}{b} \right)} \end{aligned} \right. \quad (8.5)$$

The first of these is calculated numerically and the second is recognized as an alternating geometrical series. The final result is thus

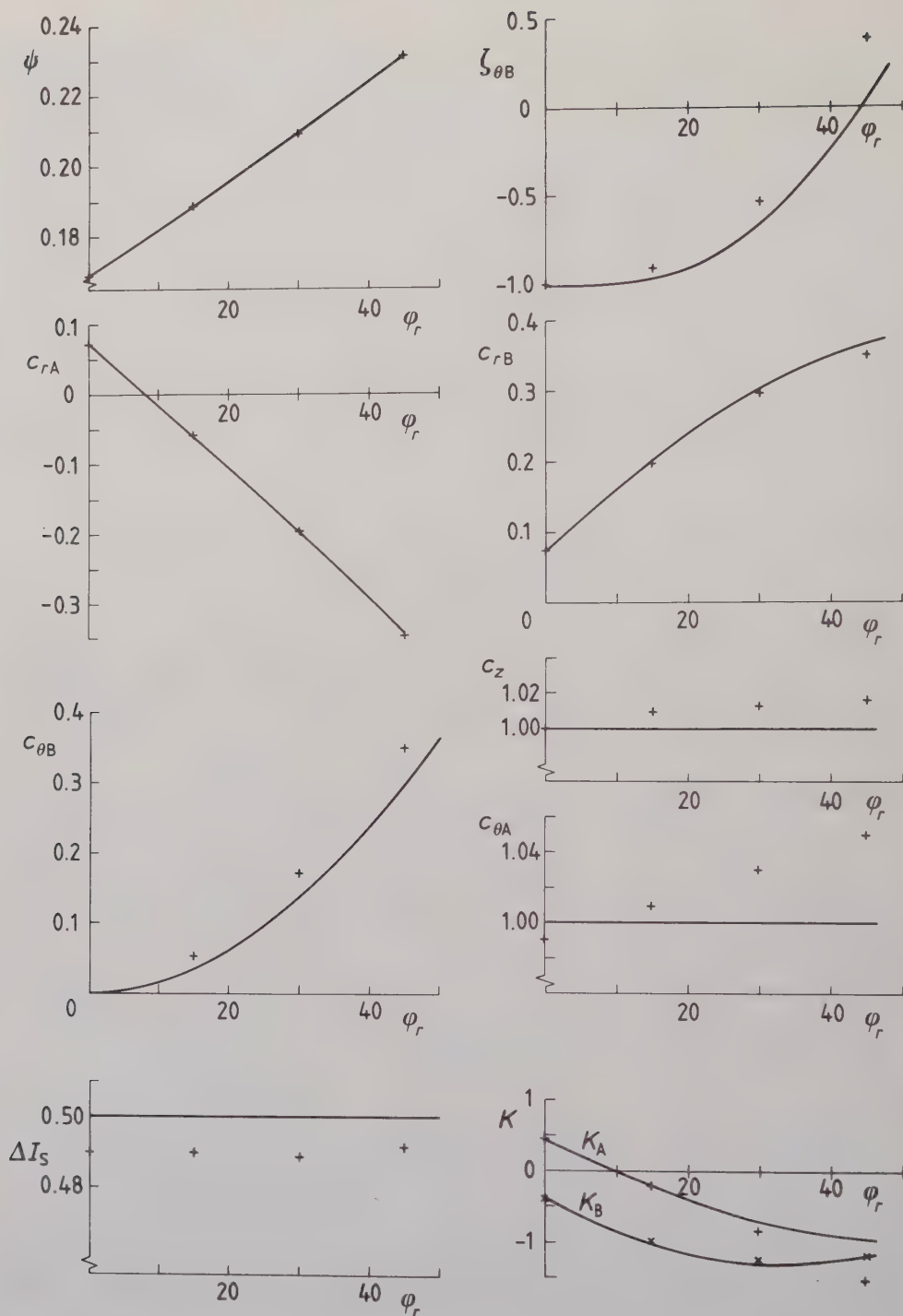


Fig. 8.5. Comparison between variations predicted, the graphs, and numerically obtained, the crosses, with respect to the blade angle φ_r for some quantities in the middle of the turbine inlet

$$\left\{ \begin{array}{l}
 \psi = 0.2 + 0.14849 \tan \frac{\varphi_r}{2} \\
 c_{rA} = -\tan \frac{\varphi_r}{2} \\
 c_{rB} = \tan \frac{\varphi_r}{2} \cos \varphi_r \\
 c_z = 1 \\
 c_{\theta B} = 1 - \cos \varphi_r \\
 \Delta I_S = 0.5 \\
 \zeta_{\theta B} = 5 \tan \frac{\varphi_r}{2} \sin^2 \varphi_r \\
 K_A = -5 \sin \frac{\varphi_r}{2} \cos \varphi_r \\
 K_B = -5 \sin \frac{\varphi_r}{2} \cos^3 \varphi_r \left(\cos^2 \frac{\varphi_r}{2} + \cos^2 \varphi_r \sin^2 \frac{\varphi_r}{2} \right)^{-3/2}
 \end{array} \right. \quad (8.6)$$

Certain constants are added to these expressions chosen to fit the numerical values for the case: $\varphi_r = 0$, when drawing the graphs in Fig. 8.5. This implies that only the changes caused by $\varphi_r \neq 0$ are proposed by Eq. (8.6). In Fig. 8.5 they are compared with the numerically obtained ones. Despite the dissimilarity between the two flow examples compared, which in fact is the reason for the disagreement in the $c_{\theta A}$ diagram as in the latter example $c_{\theta A}$ varies according to Eq. (8.3), correlation between them assuredly exists.

9. Numerical Results

Numerical calculations were carried out for the torque converter shown in Fig. 1.1. The meridional cross section used in the calculations is shown in Fig. 9.1. The measures are made non-dimensional according to Eq. (3.92) and the outer radius of the flow circuit R_D is 120 mm. The quasiorthogonal net introduced in Fig. 3.4 is generated by 27 k-lines, 9 in each turbine, shown in Fig. 9.1, and 9 i-lines which divide each k-line into 8 parts of equal length. The spaces between the turbines are considered to be small and are therefore omitted.

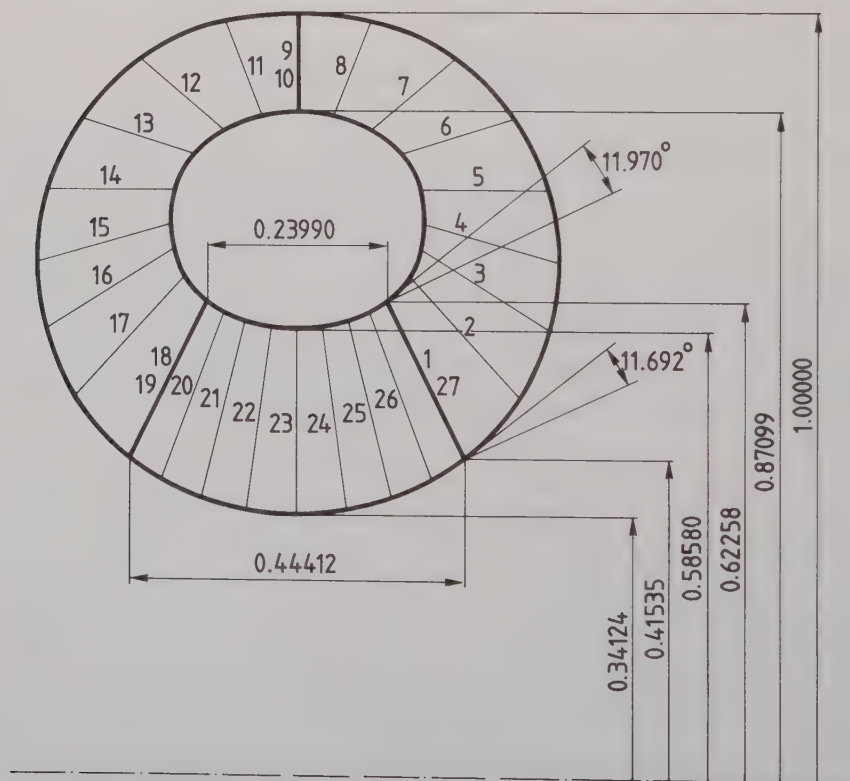


Fig. 9.1. Non-dimensional measures of the meridional cross section

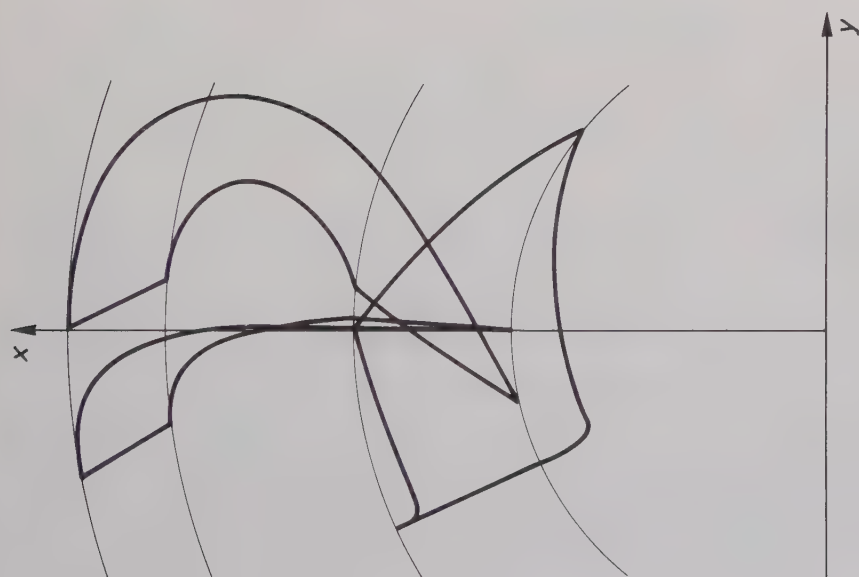


Fig. 9.2. The shape of the blades when seen in the z axis direction

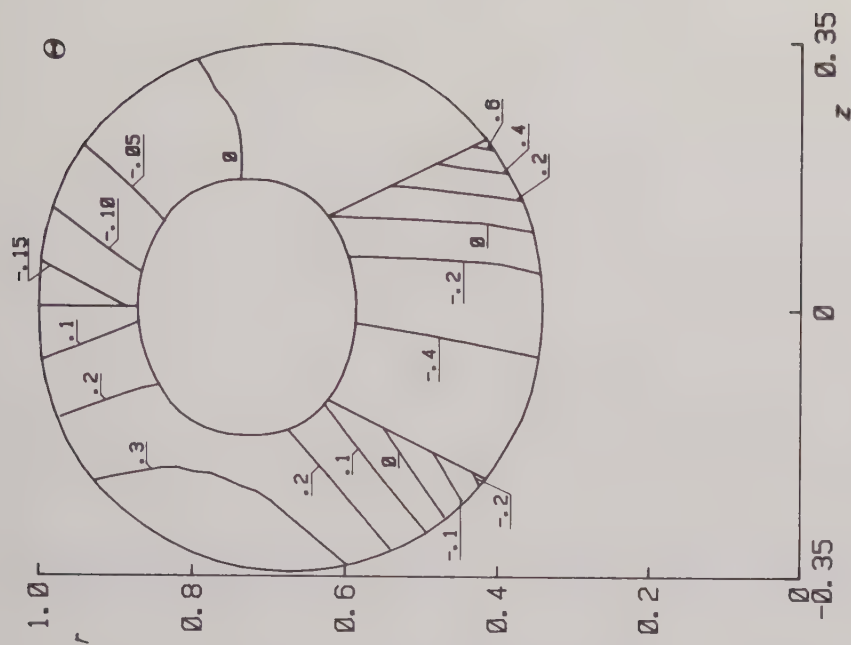


Fig. 9.3. Iso-lines for the blade surface angle Θ (radians)

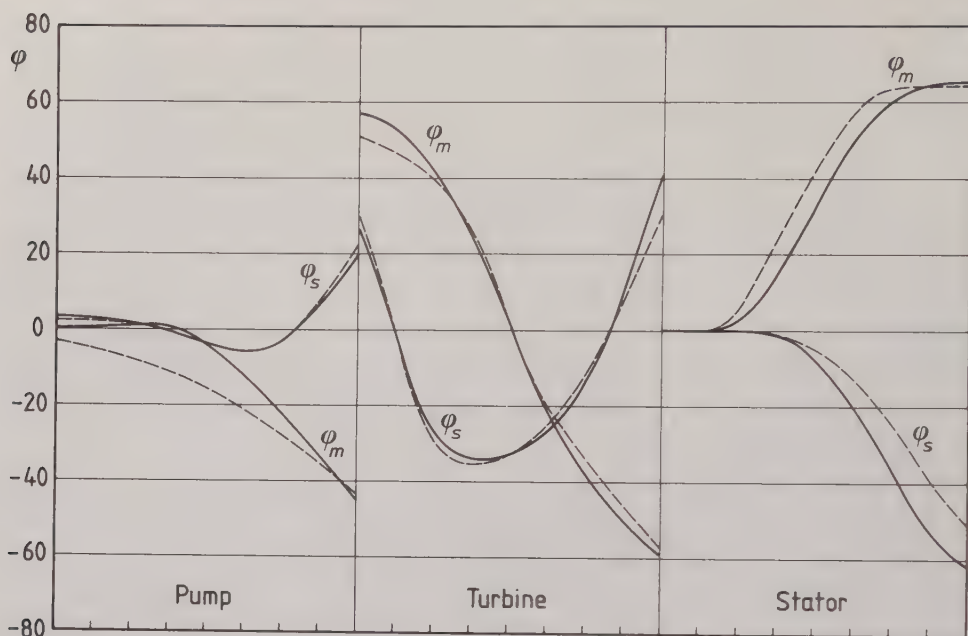


Fig. 9.4. The blade angles φ_m and φ_s at the shell, solid lines, and the core, dashed lines, versus k-line position

The 23rd k-line, in the middle of the stator, was chosen as the 0-k-line, see Sec. 7.1. Here the rothalpy and the stream function values for the traced streamlines are calculated. Nine streamlines are traced and are chosen to be those which pass the nodes on the 0-k-line.

The shapes of the blades, the blade surface angle Θ , and the blade angles φ_m and φ_s at the shell and the core are shown in Figs. 9.2–9.4. The geometries of the blades are found by placing the turbines in a milling machine and touching the blade surfaces with an awl fastened in the chuck. The coordinates are then obtained from the readings of the scales on the machine. When this is done for a number of points on the edge of the blades at the shell or the core, the function $\Theta(m)$ is obtained. The function $\varphi_m(m)$ is now calculated by Eq. (3.9) and finally a smooth curve is attached to the calculated values.

It was found that an acceptable assumption for the Θ variation along a k-line is

$$\Theta(s) = \Theta_{\text{shell}} + (\Theta_{\text{core}} - \Theta_{\text{shell}}) \frac{r_{\text{core}} s}{r s_{\text{core}}} \quad (9.1)$$

Use of Eqs. (3.9) and (3.3) now yields

$$\tan \varphi_s(s) = (\Theta_{\text{core}} - \Theta_{\text{shell}}) \frac{r_{\text{core}} r_{\text{shell}}}{r s_{\text{core}}} \quad (9.2)$$

Calculations were carried out for the angular velocities: $\omega_1 = 1$, $\omega_2 = -0.2$, 0.0, 0.2, 0.4, 0.6, 0.7, 0.8, 0.85, 0.9, 0.95, and 0.98, and $\omega_3 = 0$. The subscripts are 1, 2, 3 for the pump, turbine, and stator respectively. As the stator is mounted on a unidirectional clutch either $\omega_3 = 0$ and $M_3 > 0$ or $\omega_3 > 0$ and $M_3 = 0$ must be satisfied, see Eq. (7.34).

Some samples from the results obtained are shown below. In Figs. 9.5-9.11 the streamlines, that is the stream function Ψ , the vorticity ζ_θ , the meridional flow velocity c_m , the specific moment of momentum Γ , the force component k_θ , the pressure p , and the specific energy I are shown for the case $\omega_2 = 0.6$. As an arbitrary constant can be added to the pressure function, which is stated in Sec. 7.2, a value is chosen which gives $p = 1$ in the pump entrance at the shell.

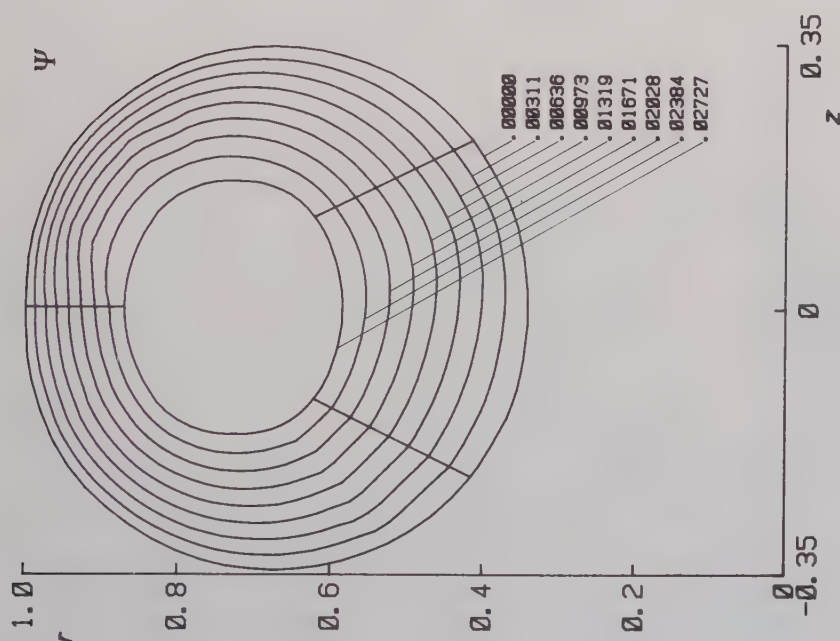


Fig. 9.5. Streamlines for the case $\omega_2 = 0.6$

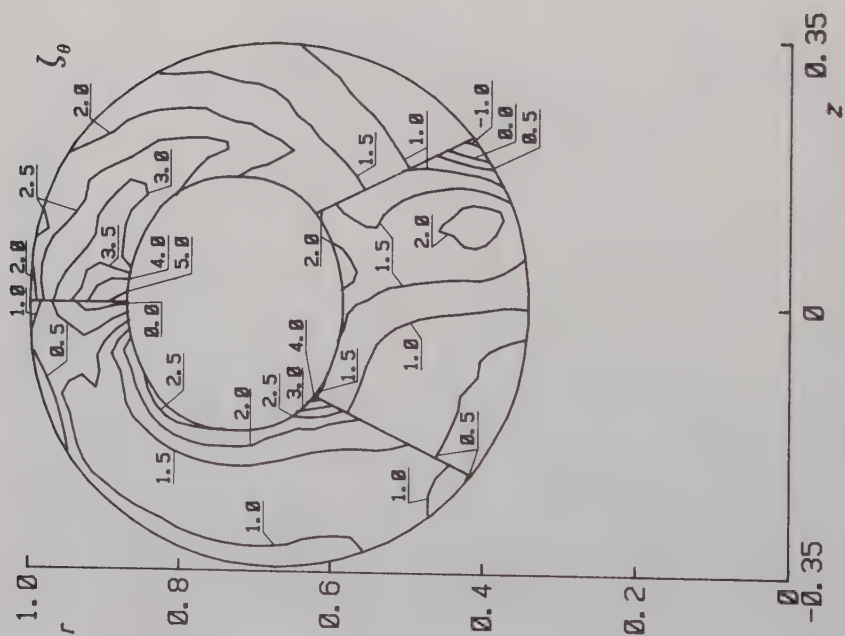


Fig. 9.6. Iso-lines for the vorticity ζ_θ for the case $\omega_2 = 0.6$

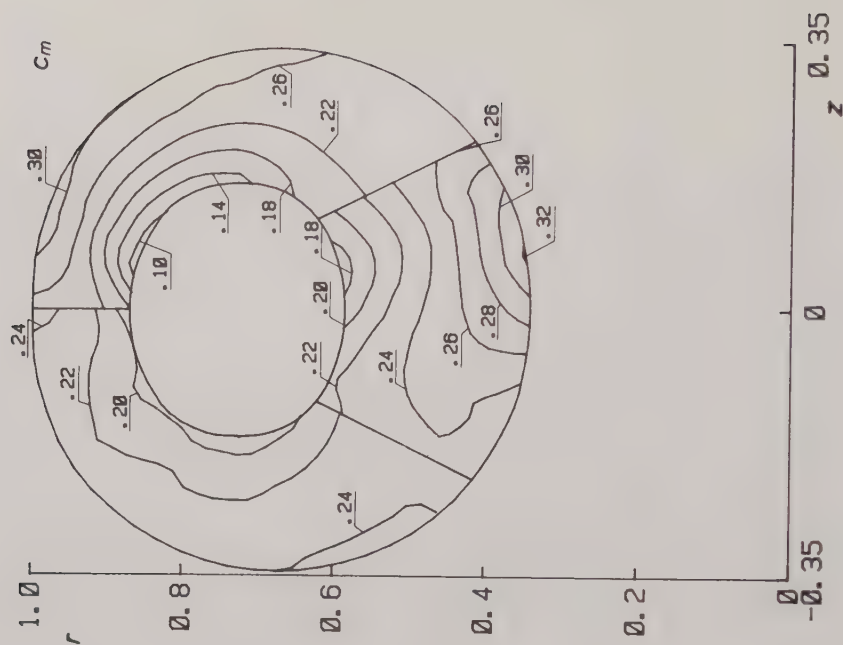


Fig. 9.7. Iso-lines for the meridional flow velocity c_m for the case $\omega_2 = 0.6$

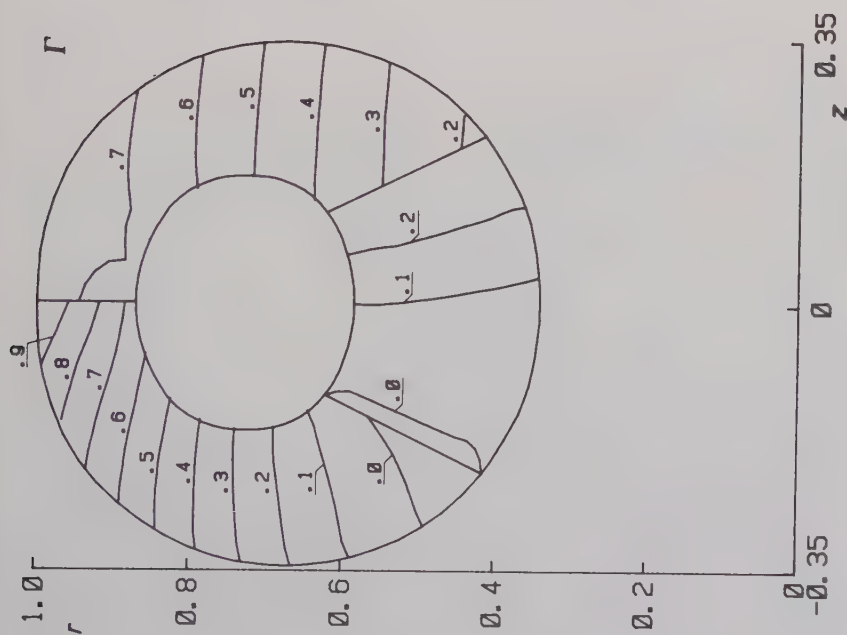


Fig. 9.8. Iso-lines for the specific moment of momentum Γ for the case $\omega_2 = 0.6$

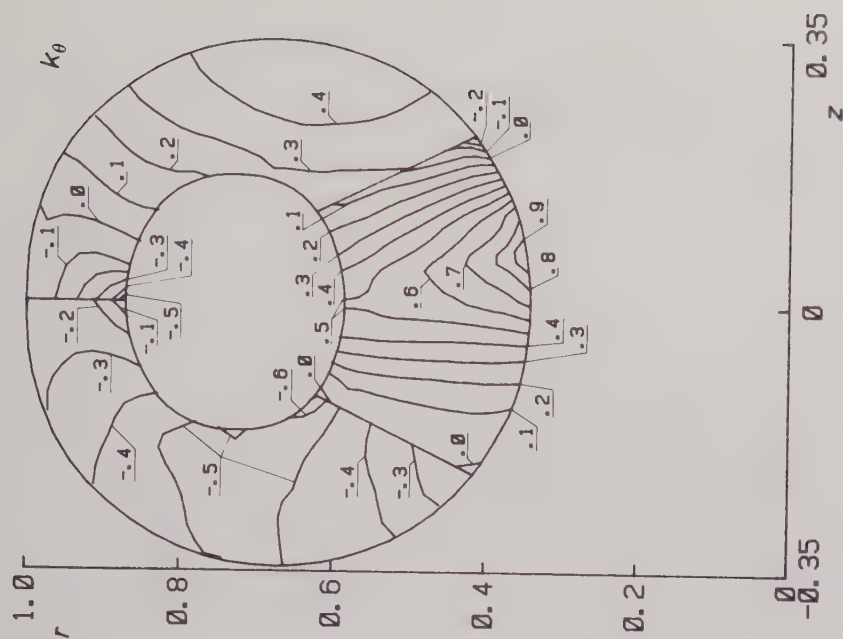


Fig. 9.9. Iso-lines for the force component k_θ for the case $\omega_2 = 0.6$

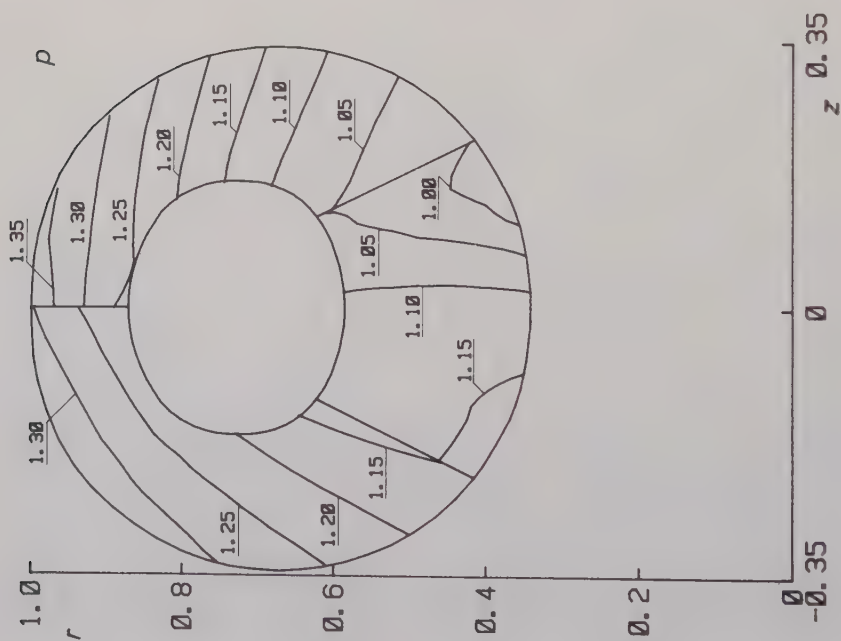


Fig. 9.10. Iso-lines for the pressure p for the case $\omega_2 = 0.6$

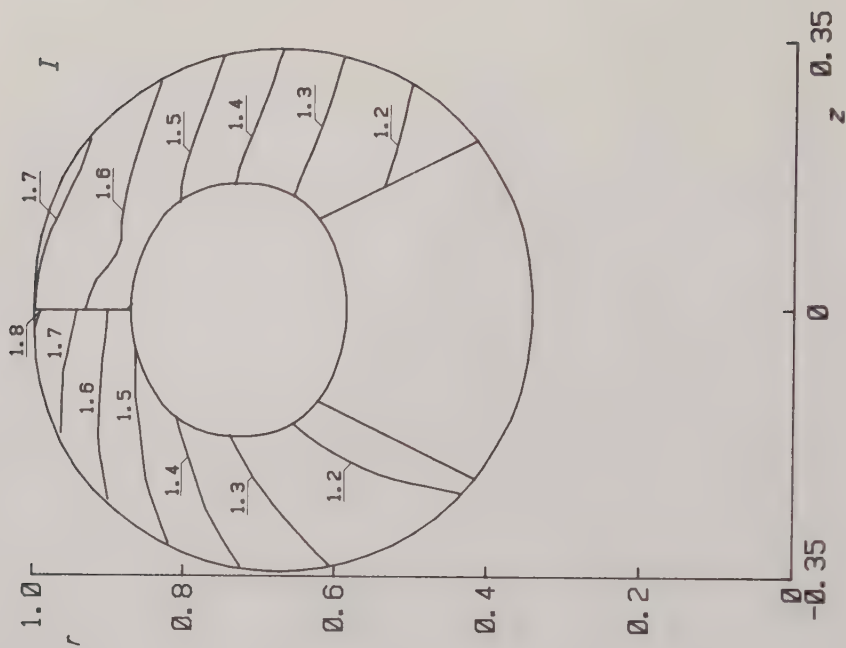
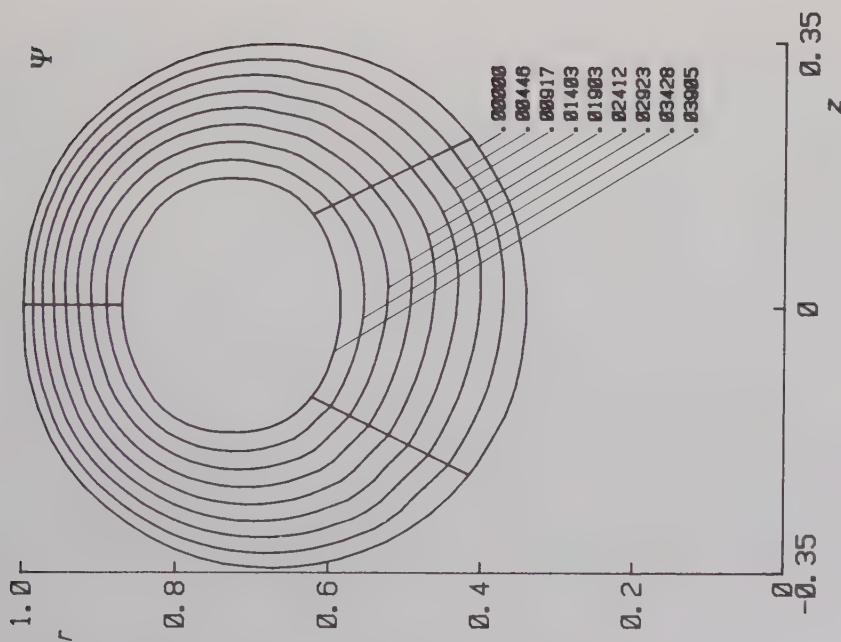
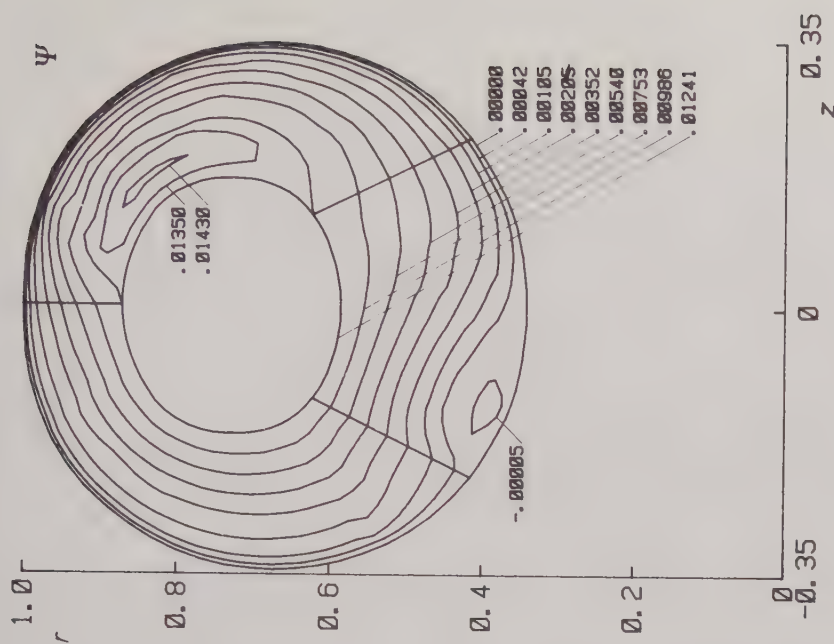


Fig. 9.11. Iso-lines for the specific energy I for the case $\omega_2 = 0.6$

Fig. 9.12. Streamlines for the case $\omega_2 = 0.2$.Fig. 9.13. Streamlines for the case $\omega_2 = 0.9$.

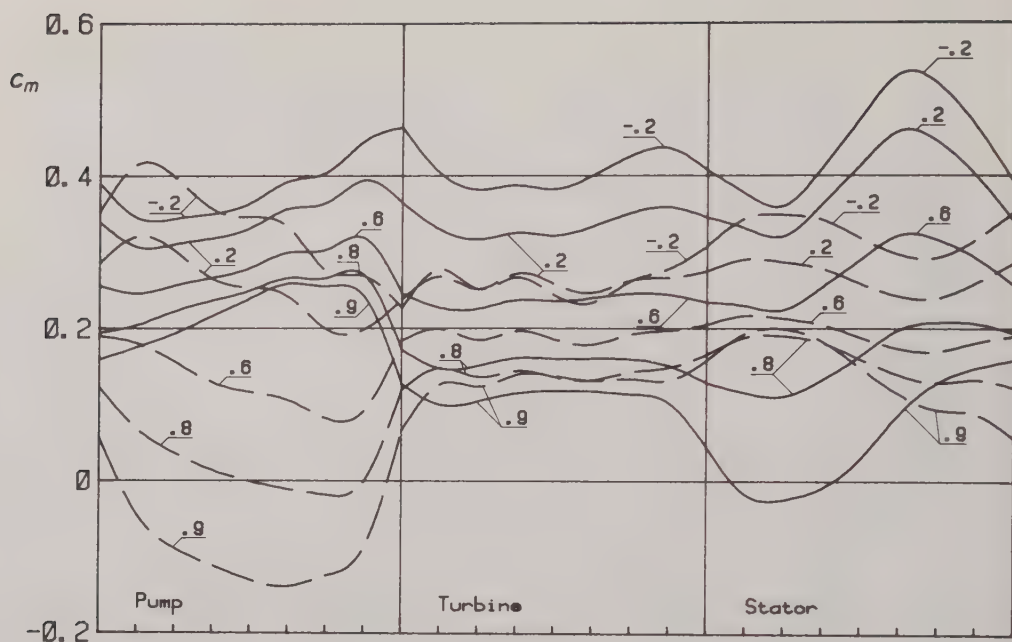


Fig. 9.14. The meridional flow velocity c_m at the shell, solid lines, and the core, dashed lines, versus k-line position for some turbine speeds, ω_2

In Figs. 9.12 and 9.13 the streamlines for the cases $\omega_2 = 0.2$ and $\omega_2 = 0.9$ are shown. The streamlines may be said to be roughly equal to the i-lines but for $\omega_2 > 0.6$ discrepancies start to develop. For $\omega_2 = 0.9$, Fig. 9.13, two eddies are obtained, one bigger at the core in the pump and a minor one at the shell in the entry part of the stator.

The meridional flow velocity c_m , the specific moment of momentum Γ , the pressure p , and the specific energy I at the shell and the core for the cases $\omega_2 = -0.2, 0.2, 0.6, 0.8$, and 0.9 are shown in Figs. 9.14–9.17. For these cases also the meridional through flow velocity c_a , the specific moment of momentum Γ , the pressure p , and the specific energy I at the inlets and outlets are shown in Figs. 9.18–9.21. In Fig. 9.22 the quantity $H(\Psi) - H(0)$ is shown in order to illustrate the rothalpy gradient $dH/d\Psi$. Some of the graphs in Figs. 9.14–9.21 were omitted if they came too close to another one, in order to

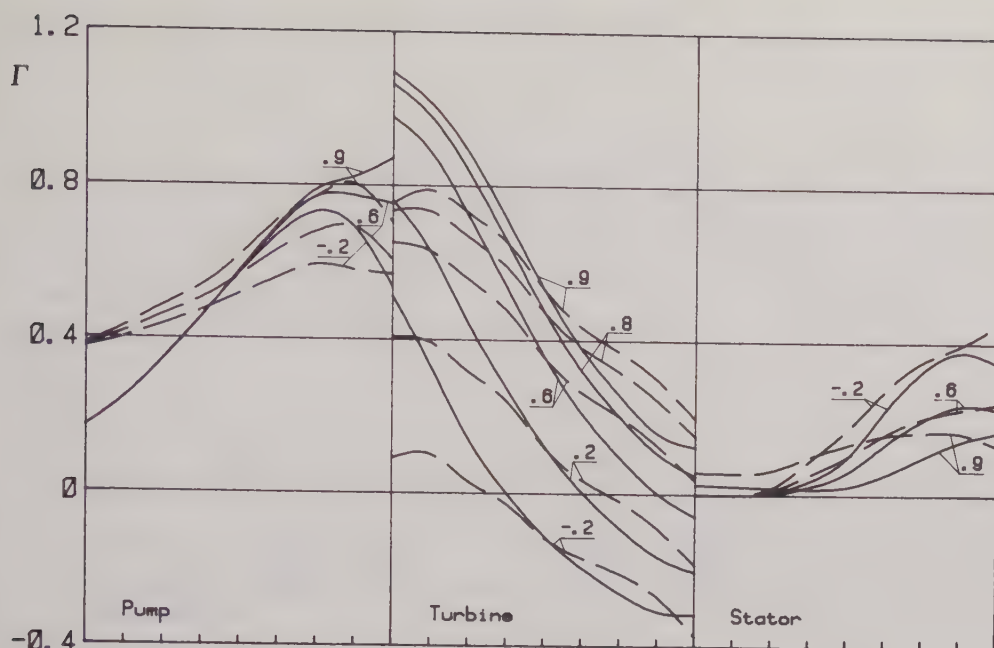


Fig. 9.15. The specific moment of momentum Γ at the shell, solid lines, and the core, dashed lines, versus k-line position for some turbine speeds, ω_2

increase the readability.

The damping coefficients used were $\Lambda = 0.1 \div 0.15$, $\Lambda_H = 0.20$, and, if ω_3 had to be calculated, $\Lambda_\omega = 1$. These values are rather low but were chosen in order to reduce the risk of failure of the calculation. Usually about 70 iterations were required and the CPU-time/iteration was 1.4 seconds (computer UNIVAC 1100).

The calculations were terminated when $|\psi_R - \psi^{(n)}|/(\psi_{\text{core}} - \psi_{\text{shell}}) < 2 \cdot 10^{-4}$. The changes of the rothalpy were insignificant during the latter part of the calculations but were noticed to reduce the convergence speed for the stream function. If the calculation of the rothalpy was inhibited the convergence speed was restored. The last few iterations were thus carried out without renewal of the rothalpy.

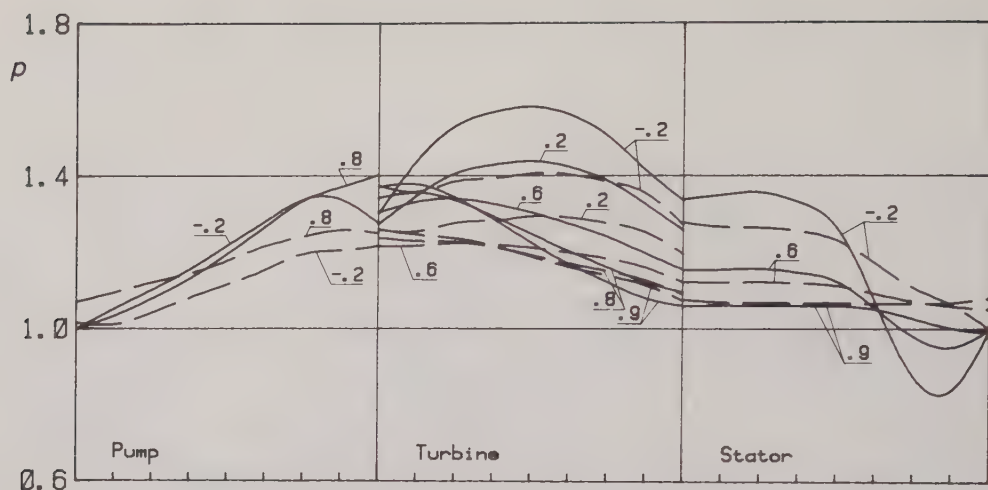


Fig. 9.16. The pressure p at the shell, solid lines, and the core, dashed lines, versus k-line position for some turbine speeds, ω_2

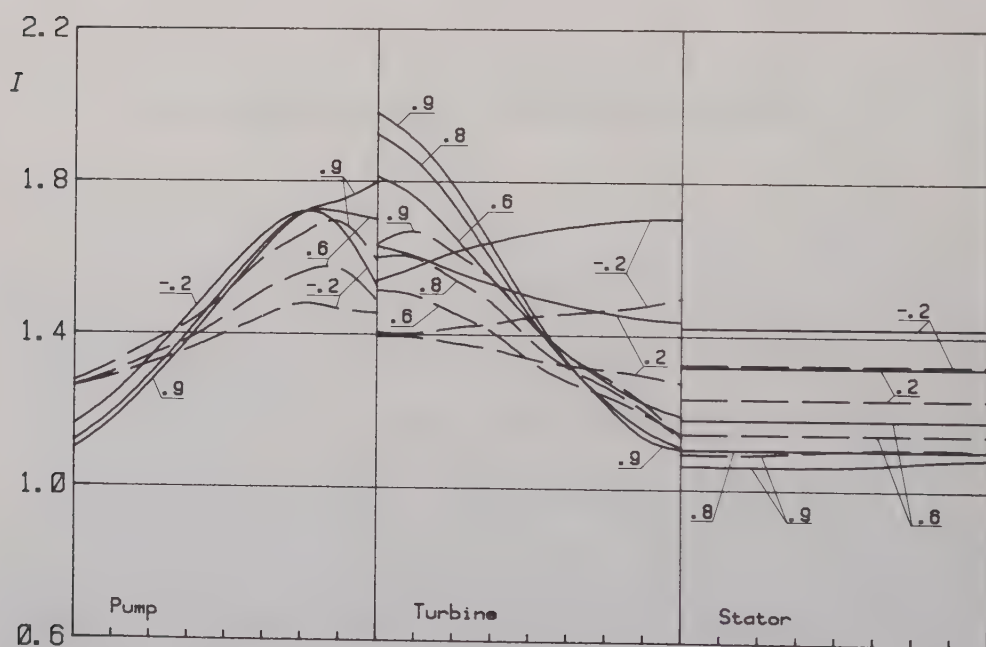


Fig. 9.17. The specific energy I at the shell, solid lines, and the core, dashed lines, versus k-line position for some turbine speeds, ω_2

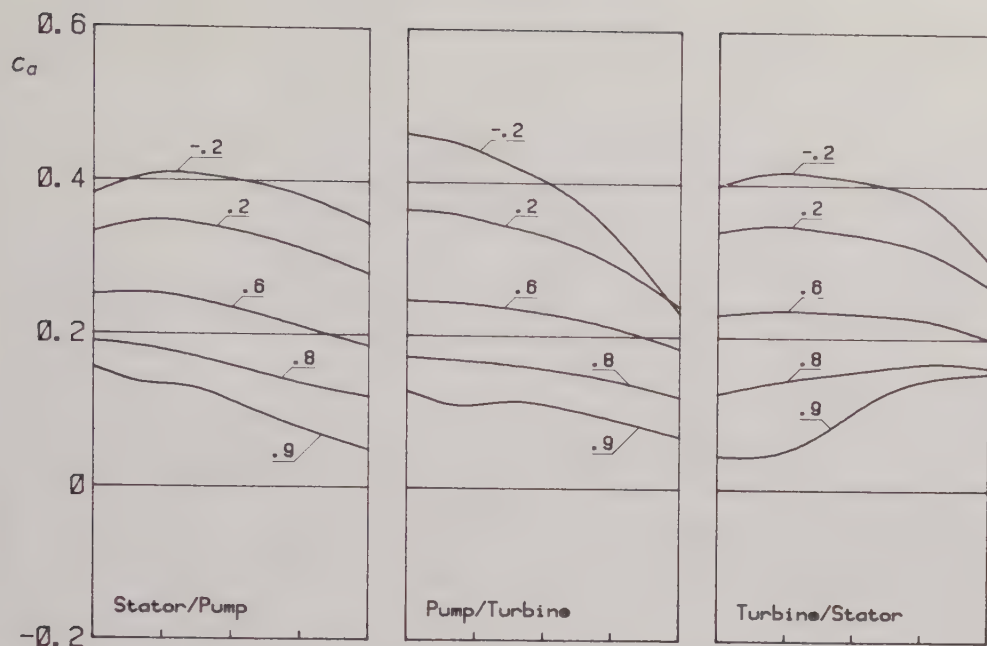


Fig. 9.18. The through flow velocity c_a at the inlets and outlets versus relative width position for some turbine speeds, ω_2

For the cases $\omega_2 = 0.95$ and 0.98 it was not possible to complete the calculation of the rothalpy as the eddies grew out over the inlets and outlets. When the eddies were within a single turbine, as in the case $\omega_2 = 0.9$, the additional rothalpy values were obtained by extrapolation of the already known ones. The incomplete calculation of the rothalpy does not entirely invalidate the result. An experiment was done with the case $\omega_2 = 0.6$, for which five iterations with $\Lambda_H = 1$ and $\Lambda = 0$, that is, with the i -lines assumed to be streamlines throughout, yielded a result in which deviation from the properly calculated one was about 30 % for the quantity $H(\Psi) - H(0)$, 4 % for circulating flow, and 6 % for the torques.

The torques for the turbines, M , and the parts of them due to the entrance shocks, M_S , are shown in Fig. 9.23. Here also the torques M_1' and M_2' calculated according to the often used one-dimensional method are shown.

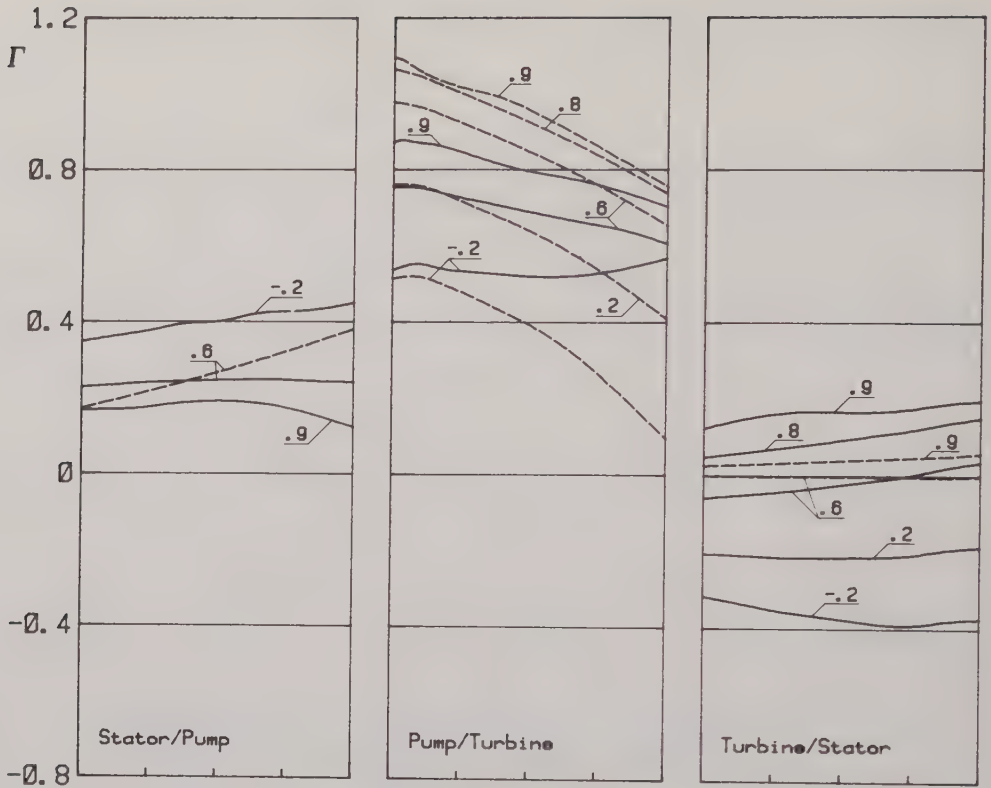


Fig. 9.19. The specific moment of momentum Γ at the outlets, solid lines, and the inlets, dashed lines, versus relative width position for some turbine speeds, ω_2

If $r_{j,2} = r_{j+1,1}$ and $\kappa_j = 1$ Eq. (7.11) reduces to

$$\left\{ \begin{array}{l} \Omega = A c_{m,0}^2 - 2 B c_{m,0} - C \\ A = \sum_{j=1}^N \left(\frac{\alpha_{j-1,2} - \alpha_{j,1}}{r_{j-1,2}} \right)^2 \\ B = \sum_{j=1}^N \omega_j (\alpha_{j+1,1} - \alpha_{j,1}) \\ C = \sum_{j=1}^N \omega_j^2 (r_{j,2}^2 - r_{j-1,2}^2) \end{array} \right. \quad (9.3)$$

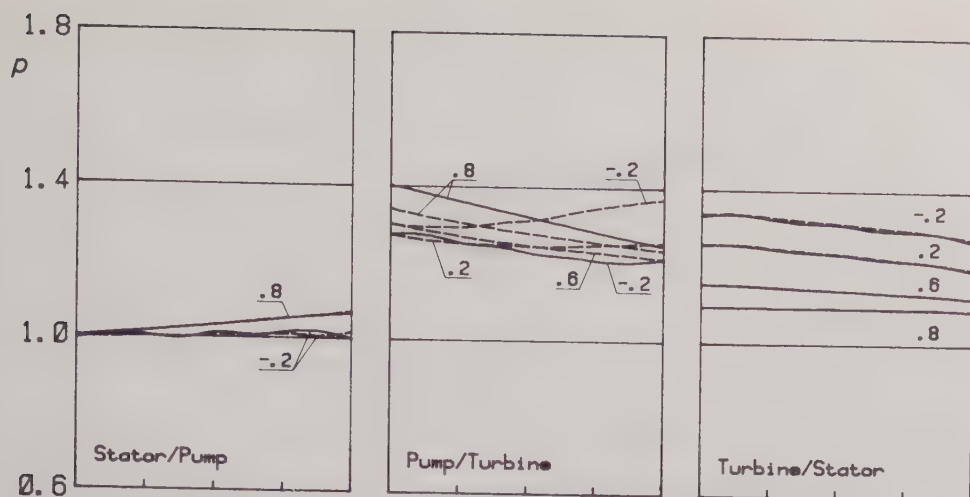


Fig. 9.20. The pressure p at the outlets, solid lines, and the inlets, dashed lines, versus relative width position for some turbine speeds, ω_2

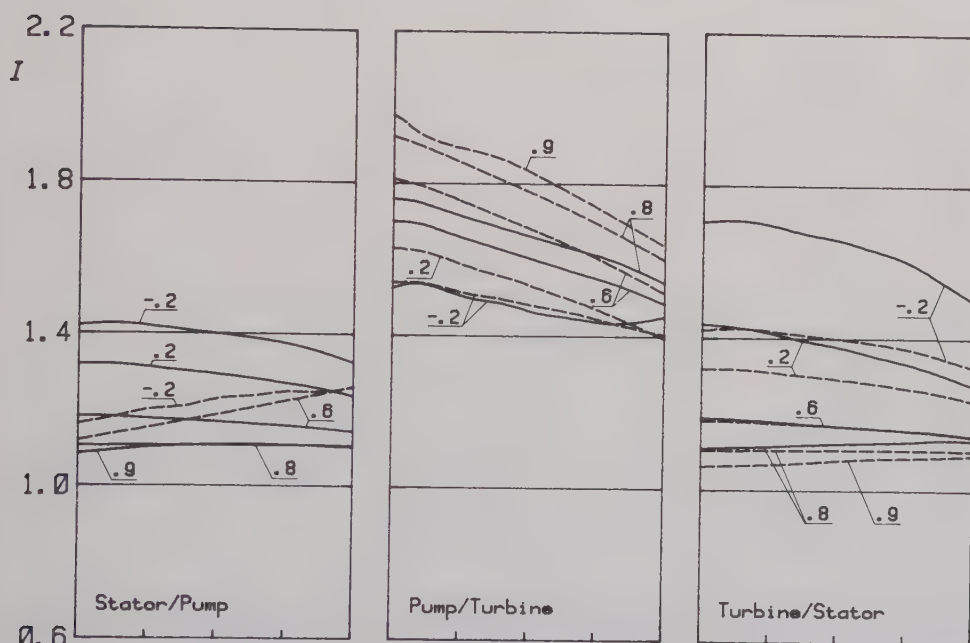


Fig. 9.21. The specific energy I at the outlets, solid lines, and the inlets, dashed lines, versus relative speeds, ω_2

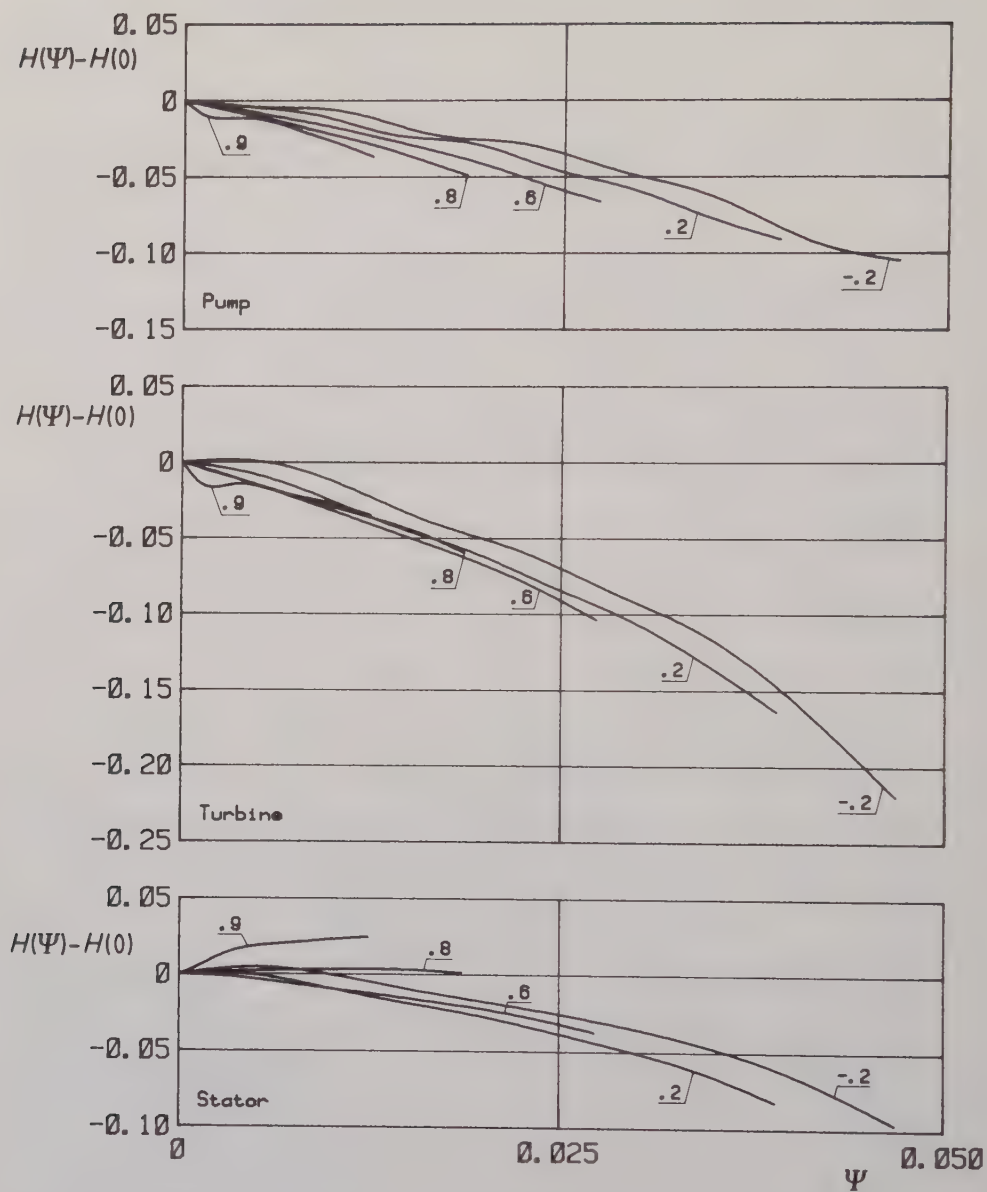


Fig. 9.22. The rothalpy functions $H(\Psi) - H(0)$ for some turbine speeds, ω_2

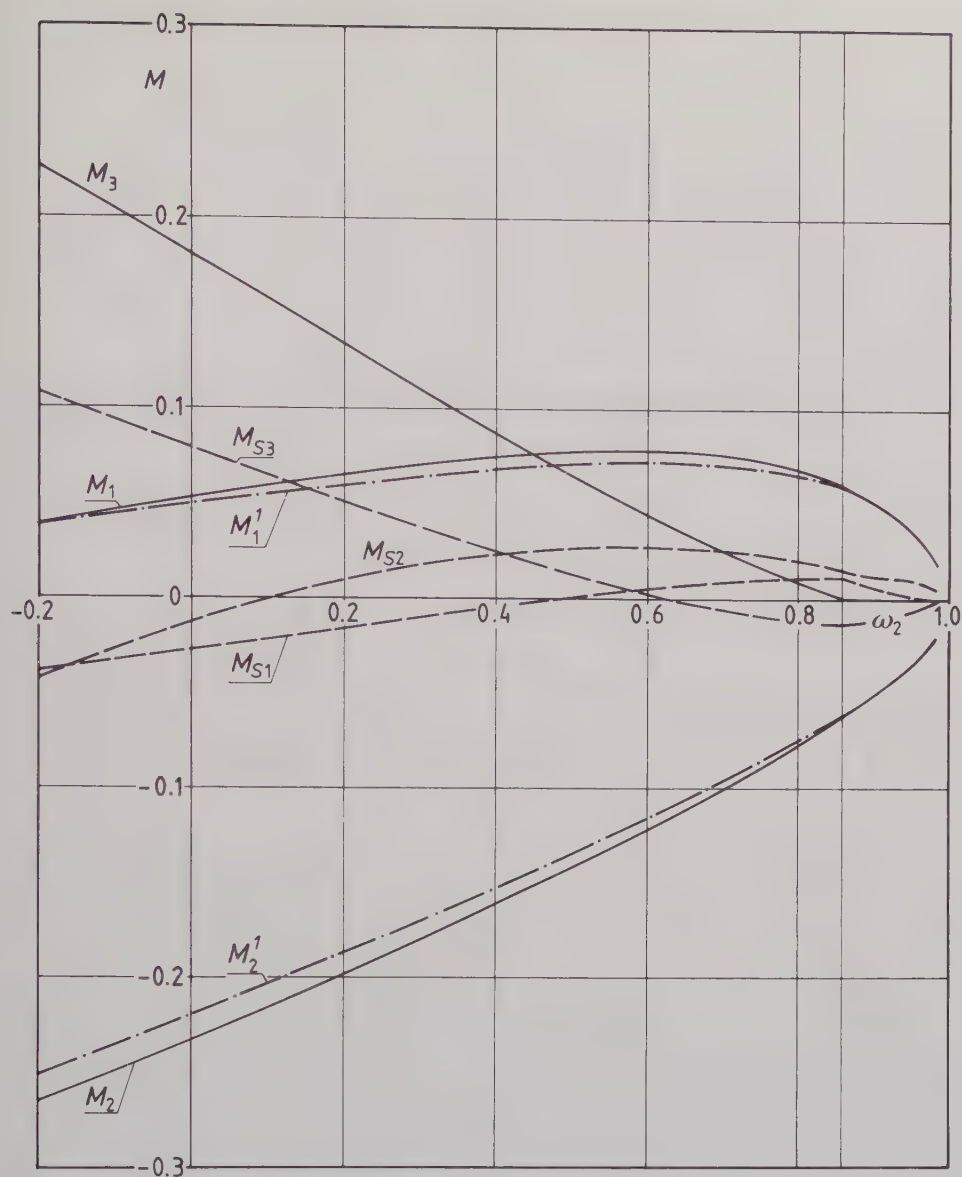


Fig. 9.23. The torques for the turbines, M , and the parts of them due to the entrance shocks, M_S , and the torques for the pump and the turbine calculated according to a one-dimensional method, M_1^I and M_2^I versus the turbine speed

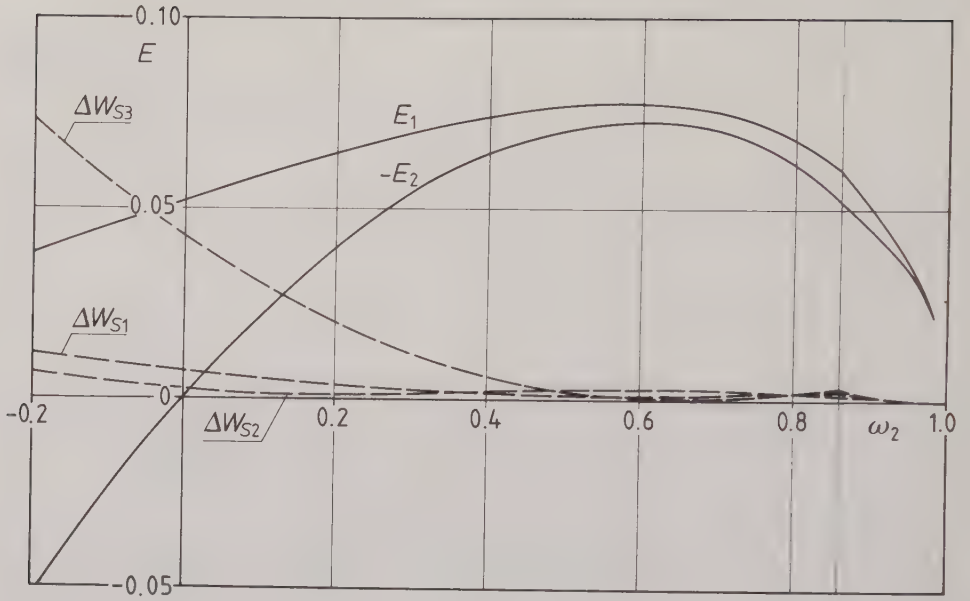


Fig. 9.24. The powers for the turbines, E , and the power losses due to the entrance shocks, ΔW_S , versus the turbine speed

A combination of Eqs. (7.38), (7.7), (3.22), and (7.43) gives the torques

$$M_j = \rho A_0 c_{m,0} (r_{j,2}^2 \omega_j - r_{j-1,2}^2 \omega_{j-1} + (\alpha_{j,2} - \alpha_{j-1,2}) c_{m,0}) \quad (9.4)$$

The coefficients were calculated according to

$$\left\{ \begin{array}{l} r = \frac{1}{2} (r_{\text{shell}} + r_{\text{core}}) \\ A = \pi r (s_{\text{core}} - s_{\text{shell}}) (\cos \lambda_{\text{shell}} + \cos \lambda_{\text{core}}) \\ a = A/A_0 \\ \alpha = \frac{1}{2} r (\tan \varphi_{m,\text{shell}} + \tan \varphi_{m,\text{core}}) / a \end{array} \right. \quad (9.5)$$

where A is a mean value for the through flow area. Use of Figs. 9.1 and 9.4 gives the values in Table 9.1.

$A_0 = 0.71225$		r	a	α
Pump	inlet	0.51897	1.03516	-0.01314
	outlet	0.93550	1.06466	-0.85626
Turbine	inlet	0.93550	1.06466	1.21907
	outlet	0.51897	1.03516	-0.80318
Stator	inlet	0.51897	1.03516	0.00000
	outlet	0.51897	1.03516	1.05150

Table 9.1. Values for the one-dimensional calculation method

In Fig. 9.24 the powers for the turbines E and the power losses ΔW_s due to the entrance shocks are shown.

The torque multiplication μ and the efficiency η are

$$\left\{ \begin{array}{l} \mu = \frac{-M_2}{M_1} \\ \eta = \frac{-E_2}{E_1} = \mu \frac{\omega_2}{\omega_1} \end{array} \right. \quad (9.6)$$

The quantities μ , η , $\dot{V}/2\pi$, ω_3 , and the area coefficient a_0 , Eq. (7.43), are shown in Fig. 9.25. Finally the values for the coefficients $\bar{r}$, $\bar{a}$, and $\bar{\alpha}$, calculated according to Eqs. (7.39), (7.40), and (7.41), are shown in Fig. 9.26. The graphs cover the torque multiplication area, $\mu > 1$, only, because the values vary greatly in the coupling area, $\mu = 1$. This is due to the development of the previously mentioned eddies which in fact constitute a flow pattern far from the assumable and desirable ones.

A final remark is that the quantity $\bar{\kappa}$, Eq. (7.42), has values about $1 \div 1.1$ when the associated power loss is significant, thus justifying the assumption $\kappa = 1$ in Eq. (9.3).

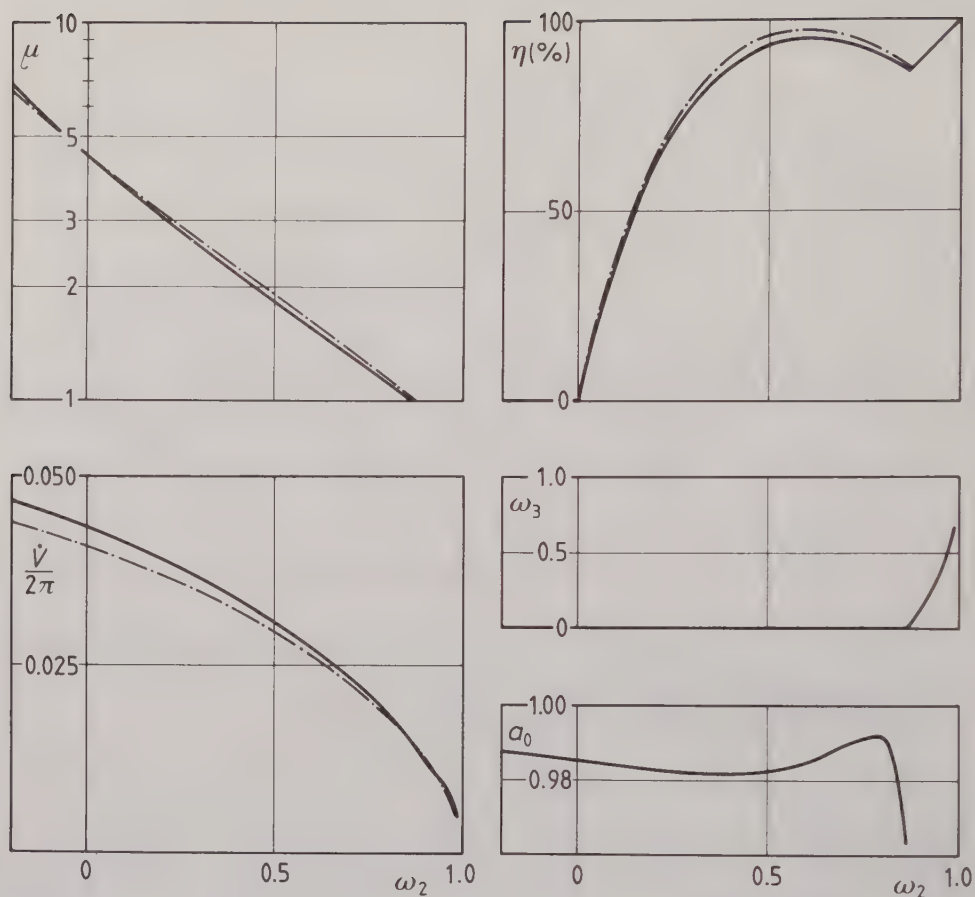


Fig. 9.25. The torque multiplication μ , the efficiency η , the circulating through flow quantity $\dot{V}/2\pi$, the stator speed ω_3 , and the area coefficient a_0 versus the turbine speed. Dash-dotted lines show results obtained by the one-dimensional method

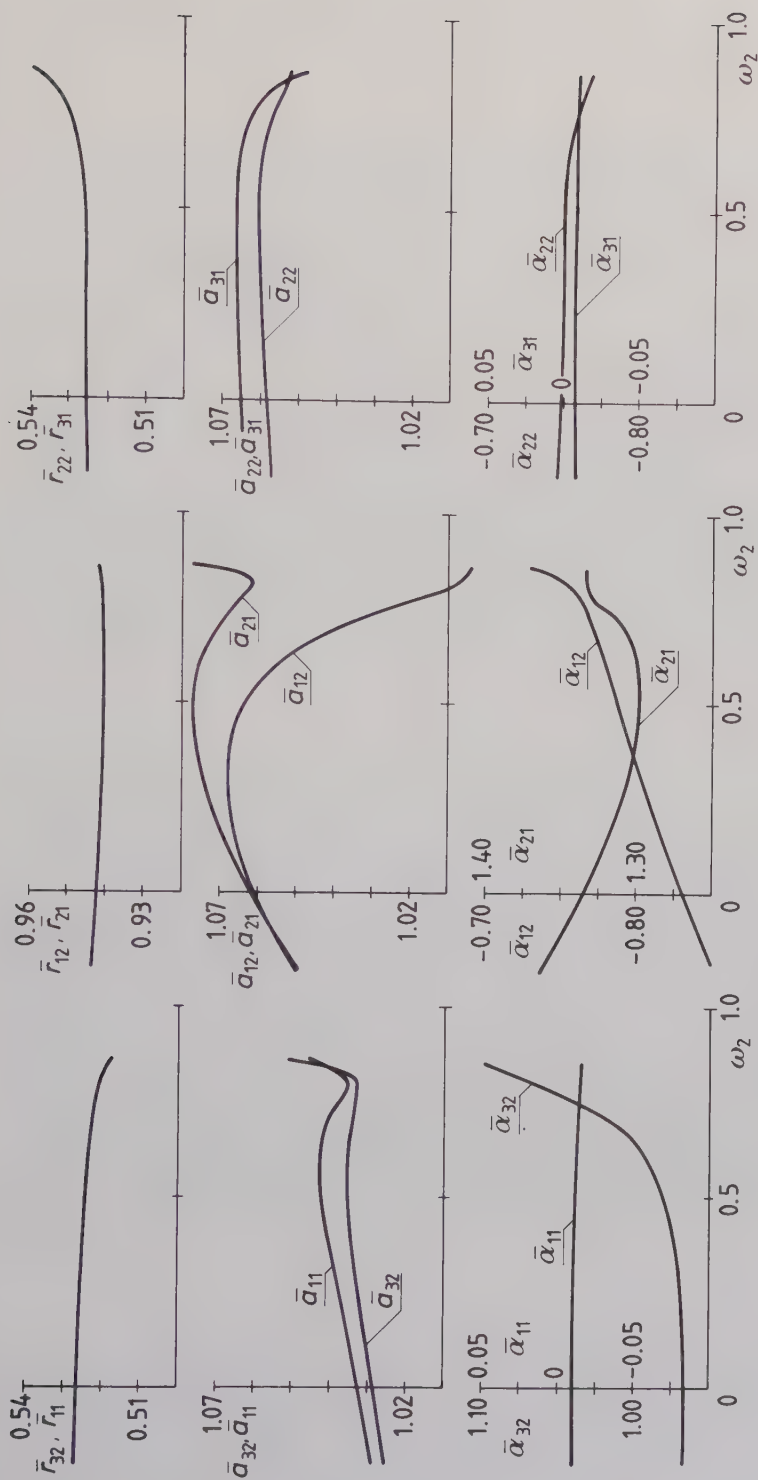


Fig. 9.26. The coefficients $\bar{r}$, $\bar{a}$, and $\bar{\alpha}$ versus the turbine speed

10. Experimental Results

A test apparatus was built in order to obtain some experimental results. Two hydrostatic motors, which can also act as pumps, having the displacements 39

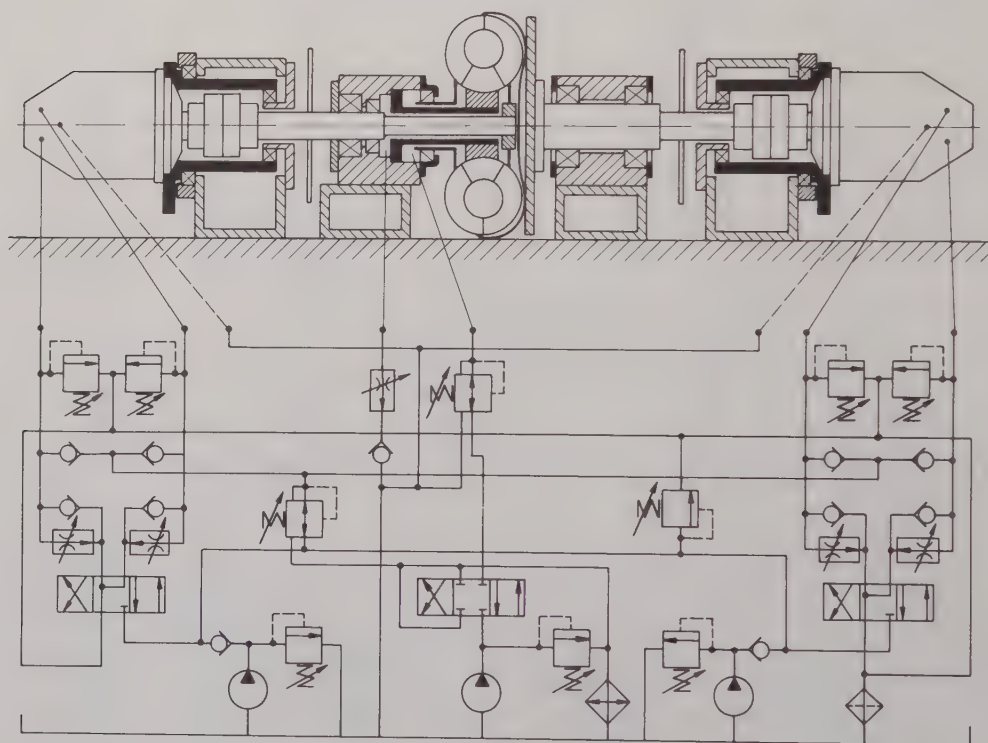


Fig. 10.1. Scheme of the test apparatus and the accessory pump machinery

and $58 \text{ cm}^3/\text{rev.}$, are connected to the pump and the turbine respectively of the torque converter. These motors are supplied with oil from pump machinery for which maximum ratings are 180 litres/min. and 140 bar for the high-pressure part. The low-pressure part also provides the charge oil to the torque converter;

charge pressure 4.5 bar. Fig. 10.1 shows a scheme of the test apparatus and the pump machinery, and Fig. 10.2 photos of the test stand and the test apparatus.

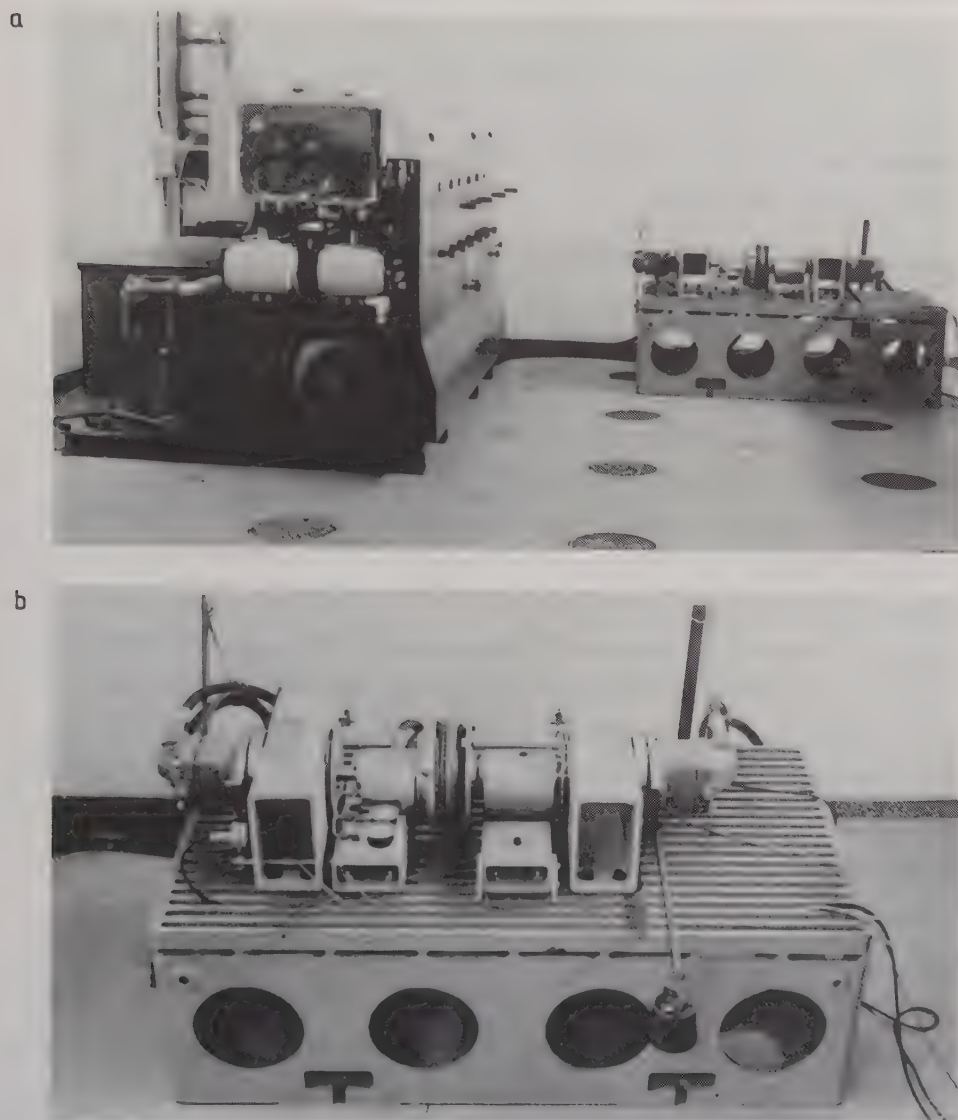


Fig. 10.2. The test stand a) and the test apparatus b)

The rotational speeds of the shafts are measured with the aid of slotted discs, photoelectric detectors, and frequency counters. The hydrostatic motors are flanged to short tubular shafts which are held stationary by force-transducers. These consist of transversally loaded rings on which strain-gauges, forming a

full bridge, are cemented. The amplification of the signals is adjusted so that the output voltage is 10 mV/Nm. The calibration is carried out by applying weights on levers fastened to the tubular shafts. The temperature of the torque converter discharge oil is measured with the aid of a NTC-resistor glued to the actual pipe.

The oil, Mobil DTE 25, has a density of 877 kg/m³ at 20°C and the dynamic viscosities 43.1 and 29.0 cSt at 40 and 50°C respectively. As the temperature decreases the density at the rate of 0.6 kg/m³°C the fluid properties may be expressed by the formulas

$$\begin{cases} \rho = 877 - 0.6 (\vartheta - 20) & (\text{kg/m}^3) \\ \nu = 29 \cdot 10^{-6} \cdot 10^{-0.0172 (\vartheta - 50)} & (\text{m}^2/\text{s}) \end{cases} \quad (10.1)$$

where ϑ denotes the temperature, ρ the density, and ν the dynamic viscosity.

A series of measurements is begun by driving both the pump and the turbine shafts forwards at equal speeds. No torque is now transmitted by the torque converter; the outputs from the torque measurements devices are caused by friction in lip seals and bearings and by static unbalances. Subtraction of these readings from the coming ones thus yields the torques due to the interior of the torque converter. Only the speed-dependent parts of the friction torques affect the results, but as a reversal of the rotational direction also reverses the direction of the friction torques, there are then greater effects on the results.

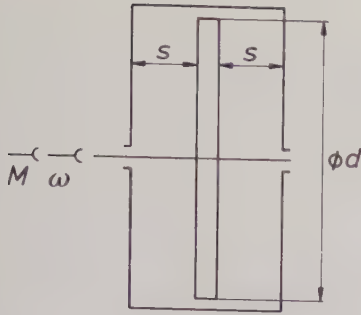
The speed of the turbine is successively decreased while the speed of the pump is kept unchanged. The registered torques for the pump and the turbine, M_P and M_T , are

$$\begin{cases} M_P = 0.93 M_1 + M_d \\ M_T = 0.93 M_2 - M_d \end{cases} \quad (10.2)$$

M_1 and M_2 are the torques for the pump and turbine introduced previously, Sec. 3.5, but as infinitesimally thin blades are presupposed there a reduction factor is used here to account for the fact that the blade material is found to occupy about 7% of the space.

The torque M_d is the drag torque between the turbine and the casing alternatively the pump. An estimate of this may be made with the aid of Ref. [5]; some quotes are shown in Fig. 10.3.

As $d = 0.24$ m, $\omega \approx 100$ rad/s, and $\nu \approx 30 \cdot 10^{-6}$ m²/s yields $Re \approx 10^5$ the



$$M = \frac{\pi}{8} \nu u \rho d^3 / s \quad Re < 3 \cdot 10^4$$

$$M = 0.472 d^3 u^2 \rho / 2 \frac{1}{\sqrt{Re}} \quad 3 \cdot 10^4 < Re < 6 \cdot 10^5$$

$$M = 0.00892 d^3 u^2 \rho / 2 \frac{1}{\sqrt[5]{Re}} \quad 6 \cdot 10^5 < Re$$

$$Re = \frac{u d}{\nu} ; u = \frac{\omega d}{2}$$

Fig. 10.3. Drag torque for a submerged rotating disc in a casing according to Ref. [5]

second of these formulas is adequate. In Fig. 10.3 both the disc and the casing are assumed to have smooth surfaces, but this is not the case for the outside of the turbine which is covered with flaps from the mounting of the blades.

According to Ref. [5] most of the oil between the disc and the casing rotates approximately as a solid body with the speed $\omega/2$, which explains the fact that the distance between the disc and the casing is not important. If this body were forced to stop the velocity gradients in the proximity of the disc would be doubled. Hence the flaps on the turbine are assumed to be taken into account by exchanging ω for 2ω in the formulas.

Insertion of $d = 2 R_D$ and $\omega = \omega_P - \omega_T$ and using Eqs. (3.93) and (3.105) gives

$$M_{0d} = 3.78 (\omega_{01} - \omega_{02}) \sqrt{\frac{\nu}{R_D^2 \omega_P} |\omega_{01} - \omega_{02}|} \quad (10.3)$$

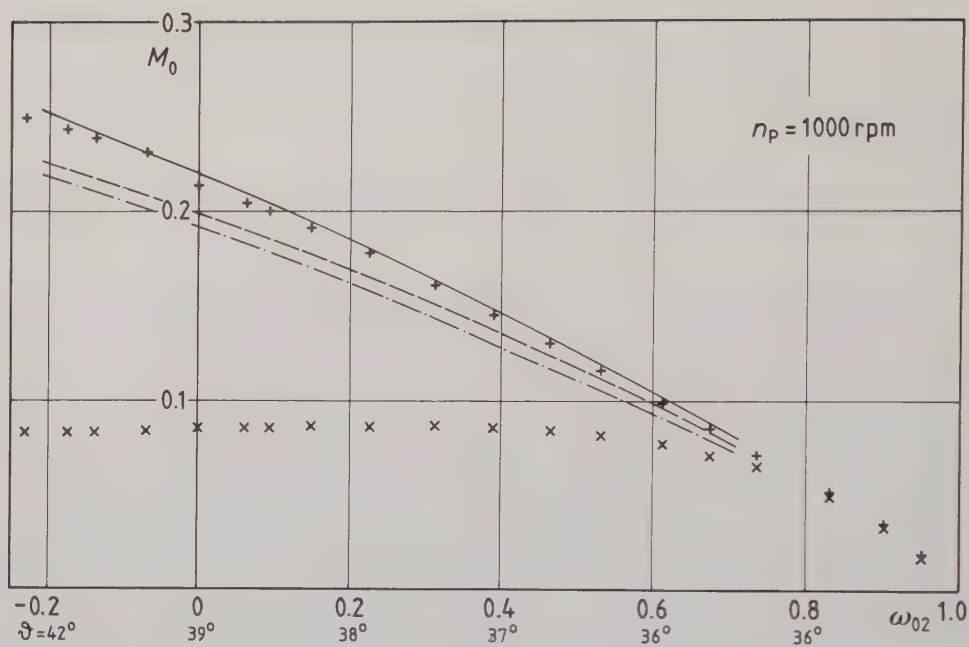
The subscript 0, denoting non-dimensional quantity is recalled in this chapter.

From the recorded quantities n_P , n_T , M_P , M_T , and ϑ , n denoting the rotational speeds in rpm, $\omega = n\pi/30$, and Eqs. (10.1), (3.93), and (3.105) the non-dimensional quantities ω_{02} , ($\omega_{01} = 1$), M_{0P} and M_{0T} are calculated ($R_D = 120$ mm).

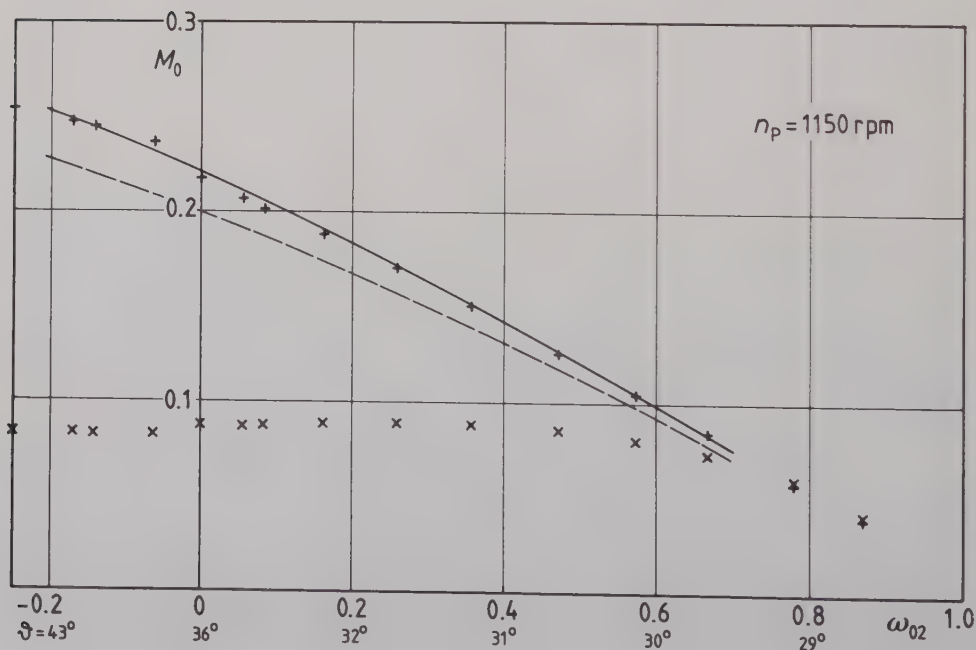
The torque quantities $M_{0P}/0.93$ and $-M_{0T}/0.93$ are plotted in Figs. 10.4. a-e. Numerous series of measurements showed that the repeatability of the tests was good.

Returning to the starting condition, that is both pump and turbine have equal speeds, after finishing a series of measurements, torques differing at most 1.5 Nm from the initially recorded ones were obtained. This quantity is assumed

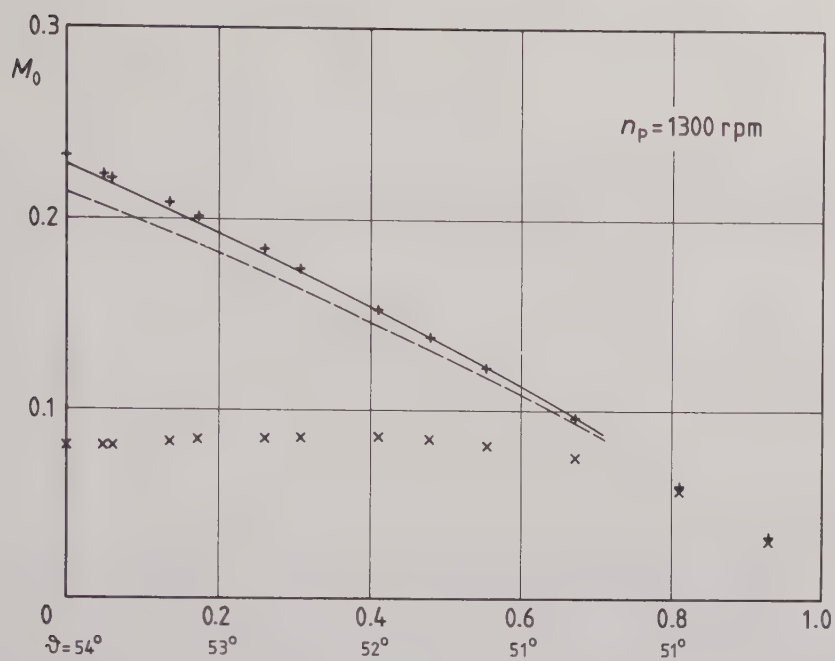
a



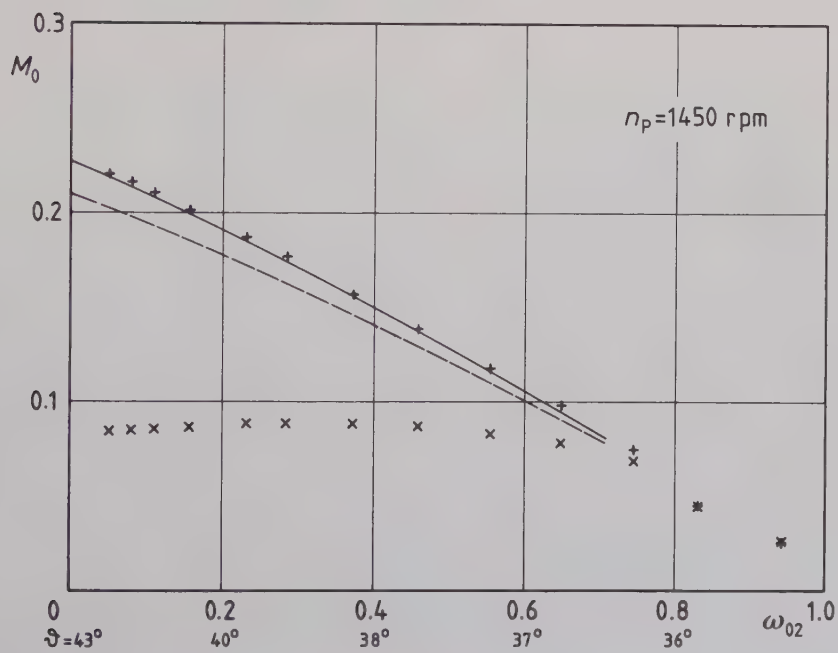
b



c



d



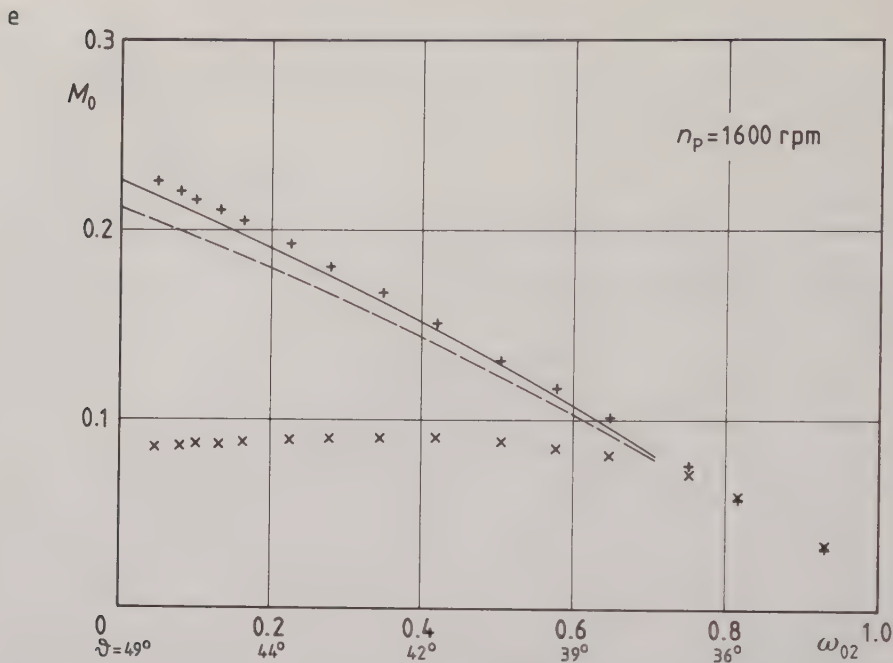


Fig. 10.4. Measured values for the quantities $M_{0P}/0.93$, ($\times$), and $-M_{0T}/0.93$, ($+$), and calculated values for $-M_{02} + M_{0d}/0.93$, solid lines, $-M_{02}$, dashed lines, and, in a), $-M_{02}^I$, dash-dotted line versus turbine speed. ϑ is the discharge oil temperature

to be a measure of the maximal error in the torque measurements. As $n_p = 1000 \div 1600 \text{ rpm}$ the non-dimensional error torque is $0.007 \div 0.003$.

The gain in the amplifier was also checked frequently in order to retain the output voltage of 10 mV/Nm .

Comparison with Fig. 9.23 shows substantial discrepancies between $M_{0P}/0.93$ and M_{01} (that is M_1 in Fig. 9.23) but the agreement between $M_{0T}/0.93$ and M_{02} is better. As the fluid friction reduces the circulating flow, that is c_m , the torque $-M_2$ is reduced too, but the torque M_1 is merely distorted; the maximum value is reached at a lower ω_{02} . When the drag torque M_d adds to $-M_2$ and M_1 , the reduction of the former is counteracted while the latter is increased all over.

As the torque equilibrium for the entire turbine system yields $\Sigma M_j = 0$ the stator torque is found, by using Eq. (10.2), to be

$$M_3 = -M_1 - M_2 = (-M_T - M_P)/0.93$$

Using Eq. (9.4) for $j = 3$ and the coefficients from Figs. 9.25 and 9.26 the mean flow velocity $\bar{c}_{0m,0}$ is calculated and used to calculate M_{02} . This process implies that the coefficients are assumed to be functions of ω_{02} only, which is assumed to be an acceptable approximation.

The torques $-M_{02}$ and $-M_{02} + M_{0d}/0.93$ are shown in Figs. 10.4. a-e, and in Fig. 10.4. a also the torque $-M_{02}^I$, which is obtained by means of the coefficients in Table 9.1. Satisfactory agreement between measured and calculated turbine torques is indicated.

A friction factor f is defined as

$$\overline{\Delta I_D} = f \frac{\bar{c}_{m,0}^2}{2} \quad (10.4)$$

Insertion of this in Eq. (7.10) yields $\bar{\Omega} = -f \bar{c}_{m,0}^2$ which, combined with Eq. (9.3), the coefficients in Fig. 9.26, and the previously calculated values for $\bar{c}_{0m,0}$, gives the f values. The calculations are carried out for $\omega_{02} = 0.7, 0.6, 0.4, 0.2, 0.0$, and, if available, -0.2 , but as experimental data are not obtained exactly for those ω_{02} values, interpolation between the existing ones is done.

In Fig. 10.5 the friction factor f is plotted against the Reynolds number

$$Re_D = \frac{\bar{c}_{m,0} R_D}{\nu} = \frac{\bar{c}_{0m,0} R_D^2 \omega_P}{\nu} \quad (10.5)$$

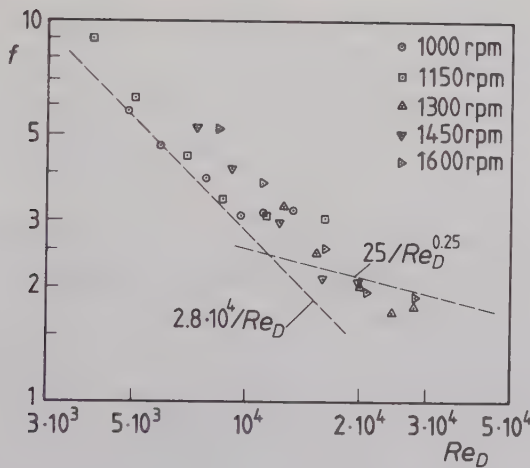


Fig. 10.5. Friction factor f versus the Reynolds number Re_D

An interesting matter is the possibility at least roughly to estimate the factor f . Eqs. (7.15) and (10.4) and Simpsons formula led to the invention of the expressions

$$\left\{ \begin{aligned} D_h &= D_{h,0} \sqrt{a} \cos \varphi_m \\ f &= \lambda_D \frac{l}{D_{h,0}} \phi_\Theta \\ \phi_\Theta &= \sum_{j=1}^N \frac{l_j}{6l} \left\{ \frac{1}{a_{j,1}^{5/2} \cos^4 \varphi_{m,j,1}} \right. \\ &\quad \left. + \frac{4 \cdot 2^{5/2}}{(a_{j,1} + a_{j,2})^{5/2} \cos^4 (\varphi_{m,j,1} + \varphi_{m,j,2})/2} + \frac{1}{a_{j,2}^{5/2} \cos^4 \varphi_{m,j,2}} \right\} \end{aligned} \right. \quad (10.6)$$

$D_{h,0}$ is a characteristic value of the hydraulic diameter, l_j a characteristic value of the length of the streamlines in the j 'th turbine, and $l = \sum l_j$. As much of the fluid friction occurs at the shell the l values are taken there, giving $l_{01} = l_{02} = 0.82$ and $l_{03} = 0.47$. A plausible value for $D_{0h,0}$ is 0.15. Insertion of the figures in Table 9.1 yields $\phi_\Theta = 3.25$ and $f = 46 \lambda_D$.

For the friction coefficient λ_D the literature, Ref. [5] among others, gives $\lambda_D = 64/Re$ for $Re < 2300$ and $\lambda_D = 0.3164/Re^{0.25}$ for $3000 < Re < 80\,000$ for smooth pipes. The first formula implies laminar flow and the second turbulent. The Reynolds number is $Re = D_h c_m / \nu$.

The factor ϕ_Θ may be regarded as a mean of the quantity $1/a^{5/2} \cos^4 \varphi_m$ and hence a mean of $\cos \varphi_m$ may be written as $1/a^{5/8} \phi_\Theta^{1/4}$ and a mean hydraulic diameter as $D_{h,0}/(a^{1/8} \phi_\Theta^{1/4})$. Insertion of the figures and Eqs. (10.5) and (7.6) gives, $\bar{a} = 1.05$, $Re = Re_D D_{0h,0}/(a^{9/8} \phi_\Theta^{1/4})$, and $f = 2.8 \cdot 10^4 / Re_D$ and $25/Re_D^{0.25}$ respectively.

Torque converters usually operate at higher pump speeds and oil temperatures than those used in these tests. This implies high values for Re_D and in this area Fig. 10.5 indicates correlation between experimental and estimated values, which leads to the suggestion

$$f = 0.316 l_0 \phi_\Theta^{1.06} D_{0h,0}^{-1.25} a^{0.28} / Re_D^{0.25} \quad (10.7)$$

Often, for instance in Ref. [8], a constant value of f is used.

Finally an exploration of other modes of operation of the torque converter was undertaken. One of the shafts was rotated with 1000 or -1000 rpm and the

other with various speeds within the interval $[-1000, 1000]$ rpm.

The result is shown in Figs. 10.6 and 10.7. In these diagrams the gradation of the ω_{02} axis is inverse for $|\omega_{02}| > 1$, that is, linear with respect to the quantity $1/\omega_{02}$, which may also be written as n_P/n_T .

Calculated values, here using the one-dimensional method, are also drawn in these diagrams. The factor f and the drag torque M_d are now determined so that measured and calculated values coincide for $n_P = 1000$ and $n_T = 0$ rpm; a principle which is sometimes used, for instance by Ref. [24]. This leads to $f = 2.56$ for $Re_D = 1.05 \cdot 10^4$ and to the conclusion that the value from Eq. (10.3) should be multiplied by 1.04. From Fig. 10.5 it appears that $f \sim Re_D^{-0.6}$ may be a suitable approximation and therefore the formula $f = 6.6 \cdot 10^2 Re_D^{-0.6}$ was used.

In the vicinity of the cases $n_P = 1000$, $n_T = -1000$ and $n_P = -1000$, $n_T = \pm 1000$ rpm discontinuity in the operation of the torque converter is expected due to the reversal of the flow in it. Only in the first case were distinct switches noticed between two modes of operation.

The coincidence between measured and predicted values was found to have deteriorated for some modes of operation, which, however, all are far from those the machine is designed for and also rarely used. For the case $n_P = 1000$ rpm the friction factor f was calculated for $\omega_{02} = -0.4, -0.6, -0.8$, and -1.0 using the coefficients at $\omega_{02} = -0.2$ in Fig. 9.26. These values and those previously obtained, Fig. 10.5, are shown in Fig. 10.8 together with the reciprocal of the Reynolds number, $1/Re_D$, and the quantity: $\sqrt{2 \Delta I_S / \bar{c}_{m,0}^2}$. This is obtained from Eqs. (7.9), (7.10), (10.4), (7.38), (3.22), and (7.43)

$$\frac{2 \Delta I_S}{\bar{c}_{m,0}^2} = \frac{2}{\rho A_0 a_0 \bar{c}_{m,0}^3} \sum_{j=1}^N M_j \omega_j - f \quad (10.8)$$

and may be regarded as a measure of the shock intensity, see Eq. (4.34). Fig. 10.8 indicates that f is correlated to both the Reynolds number and the shock intensity. The increase of f for negative ω_{02} may be caused by additional losses due to secondary flows. The situation may be likened to the flow through sharp elbow pipes. For such cases Ref. [5] gives loss coefficients which increase considerably as the deflection angle increases.

A check on the effect of the pressure level in the torque converter was also carried out. The charge pressure was varied within the interval $3 \div 6$ bar and was found to give absolutely no changes of the torques.

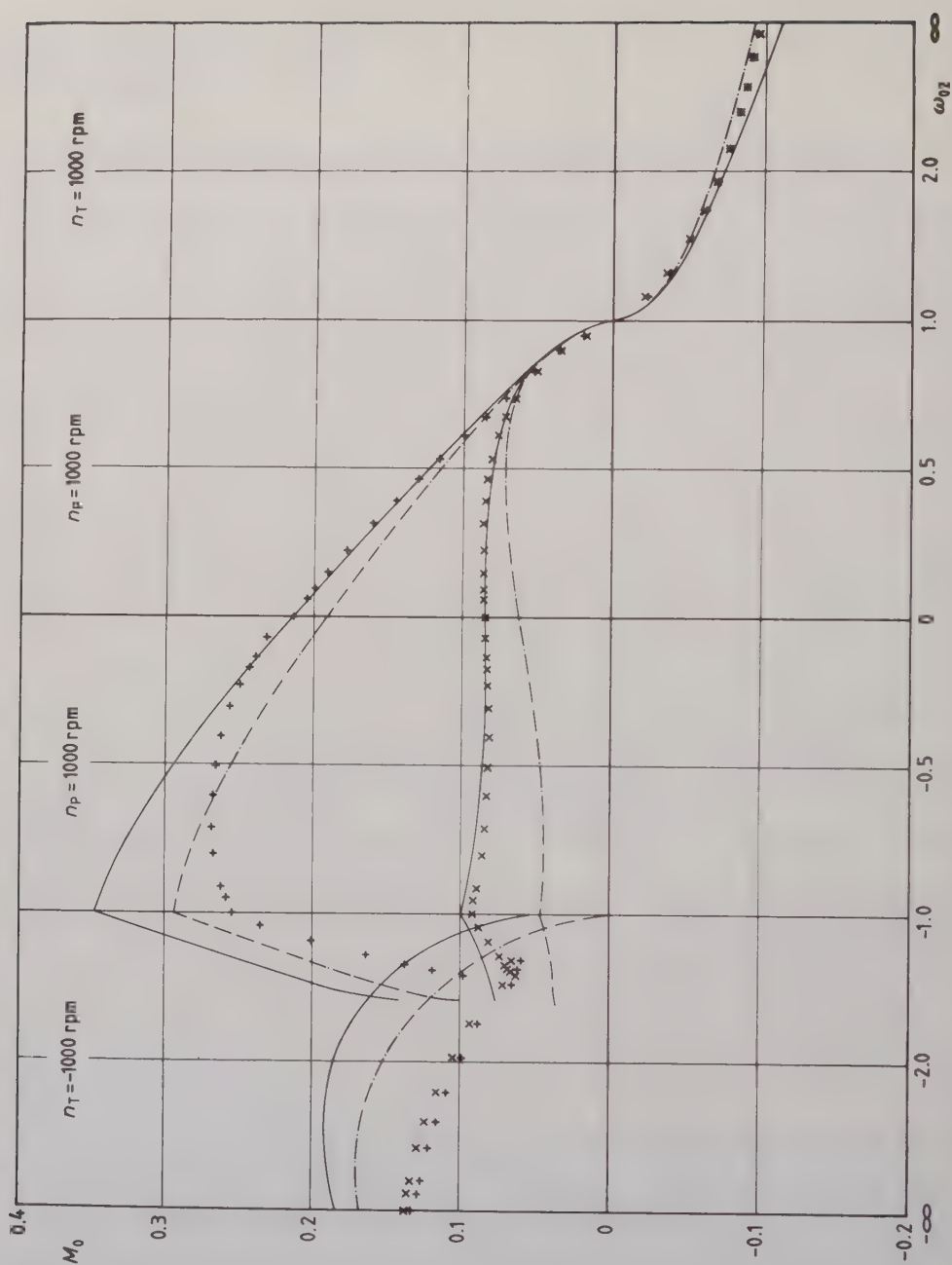


Fig. 10.6. Measured values for the quantities $M_{0P}/0.93$, ($\times$), and $-M_{0T}/0.93$, ($+$), and calculated values for $M_{01}^I + M_{0d}/0.93$ and $-M_{02}^I + M_{0d}/0.93$, solid lines, and M_{01}^I and $-M_{02}^I$, dash-dotted lines, versus turbine speed ω_{02} for $n_p \geq 0$

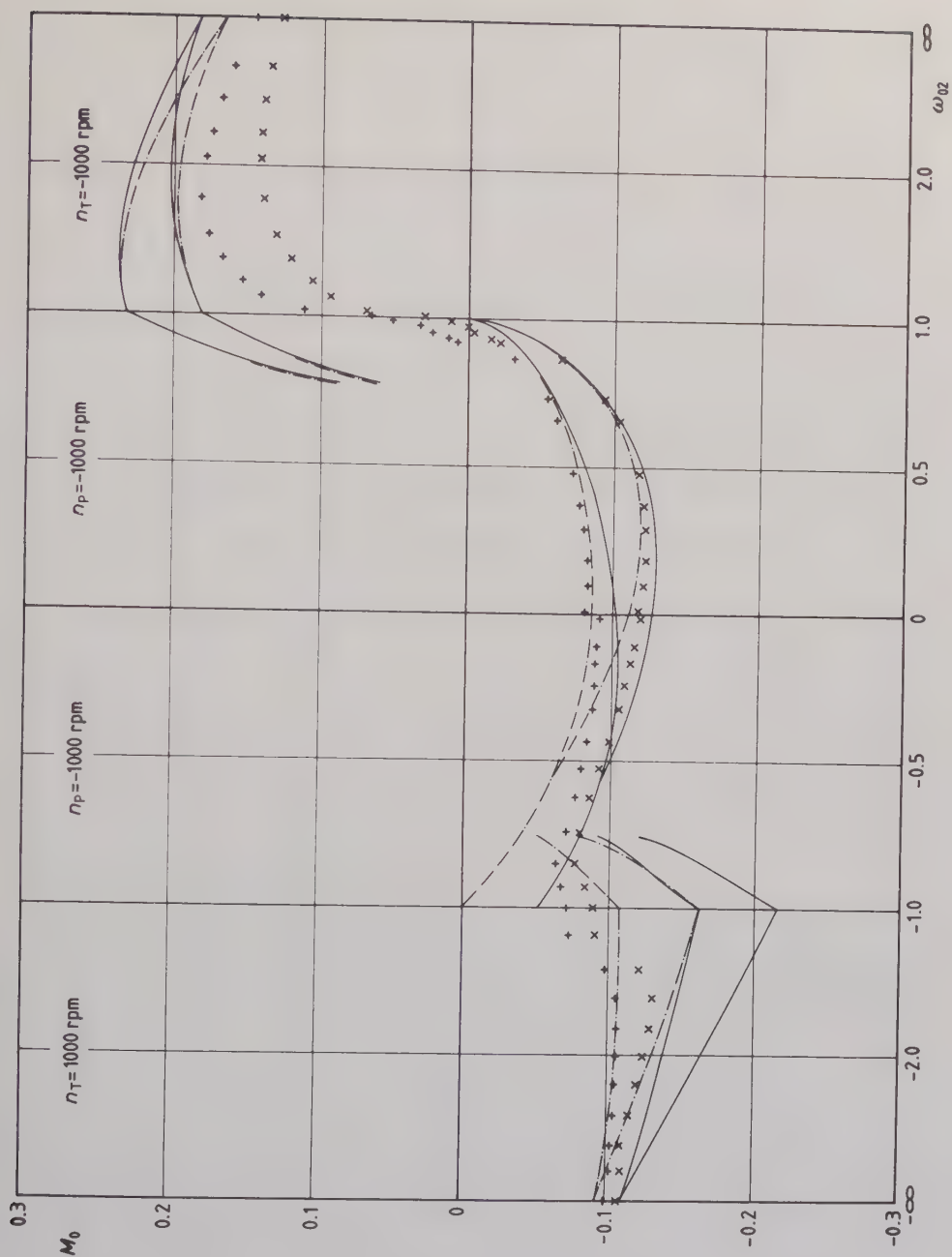


Fig. 10.7. Measured values for the quantities $M_{0P}/0.93$, ($\times$), and $-M_{0T}/0.93$, ($+$), and calculated values for $M_{01}^I + M_{0d}/0.93$ and $-M_{02}^I + M_{0d}/0.93$, solid lines, and M_{01}^I and $-M_{02}^I$, dash-dotted lines, versus turbine speed ω_{02} for $n_P \leq 0$

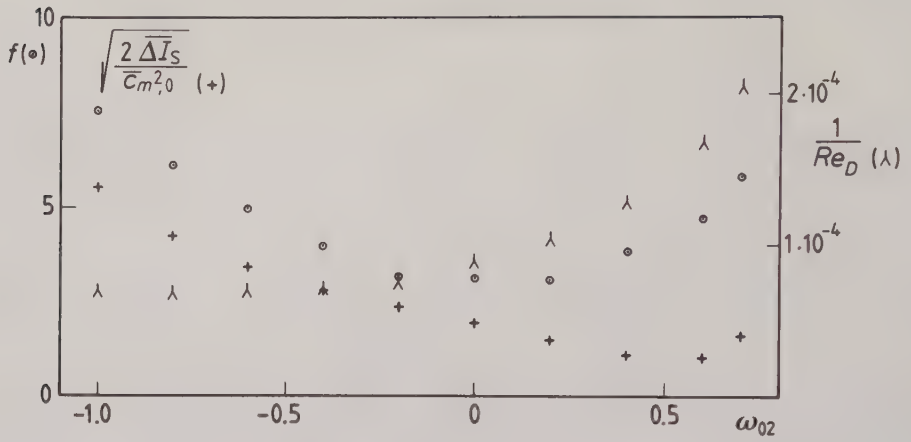


Fig. 10.8. Friction factor f , shock intensity measure $\sqrt{2 \Delta \bar{I}_S / \bar{c}_{m,0}^2}$, and reciprocal Reynolds number $1/Re_D$ versus turbine speed for $n_p = 1000$ rpm

11. Conclusion

Analysis of hydrodynamic torque converters is the subject of this report. The flow is assumed to be two-dimensional, that is no variation in the circumferential direction, frictionless, and stationary. Infinitely thin blades and constant fluid density are also assumed. Governing equations and applicable numerical techniques are examined and numerical results are obtained for the three-element torque converter W 240 (Fichtel & Sachs). The torques thus obtained appeared to be only slightly greater, at most 10% for the pump and 6% for the turbine, than the torques obtained from the frequently used one-dimensional theory.

Some experiments were carried out with the actual torque converter. Measured and calculated turbine torques were compared and it was found that the two-dimensional theory produced satisfactory values, a little better than the one-dimensional one. A friction factor which accounts for the dissipation of real flow was also evaluated. An exploration of other modes of operation, including backwards rotation of pump and turbine, showed that satisfactory prediction of these is possible in some cases.

The final conclusion is that the commonly used one-dimensional theory is useful when analysing the performance of a torque converter, whilst the two-dimensional theory is relevant when the flow pattern in it is the subject of interest.

12. References

1. Adler, D., and Krimerman, Y.: The Numerical Calculation of the Meridional Flow Field in Turbomachines Using the Finite Elements Method. *Israel Journal of Technology*, Vol. 12, 1974, pp. 268-274.
2. Chung, T. J.: *Finite Element Analysis in Fluid Dynamics*. McGraw-Hill, New York, 1978.
3. Davis, W.R., and Millar, D. A. J.: A Comparison of the Matrix and Streamline Curvature Methods of Axial Flow Turbomachinery Analysis, From a User's Point of View. *Journal of Engineering for Power*, Trans. ASME, Oct. 1975, pp. 549-560.
4. Diederichs, M.: *Innere Leistungsverzweigung durch Föttinger-Wandler*. Diss. TH Karlsruhe, 1956.
5. Eck, B.: *Technische Strömungslehre*. Springer-Verlag, Berlin, 1961.
6. Forsythe, G. E., and Warsow, W. R.: *Finite-Difference Methods for Partial Differential Equations*. John Wiley & Sons, New York, 1964.
7. Gruber, J., and Vajna, Z.: Approximative Process for Dimensioning of Hydrodynamic Torque Converters. *Acta Technica Academiae Scientiarum Hungaricae*, Tomus 64 (3-4), 1969, pp. 455-470.
8. Herbertz, R.: *Untersuchung des dynamischen Verhaltens von Föttinger-Getrieben*, Diss. TH Hannover, 1973.
Also published as an article in: *Konstruktion* 25, 1973, pp. 441-446.
9. Hirsch, Ch., and Warzee, G.: A Finite-Element Method for Through Flow Calculations in Turbomachines. *Journal of Fluids Engineering*, Trans. ASME, Sept. 1976, pp. 403-421.
10. Horlock, J. H., and Marsh, H.: Flow Models for Turbomachines. *Journal Mechanical Engineering Science*, Vol. 13, No. 5, 1971, pp. 358-368.
11. Iten, O.: *Anwendung der Methode der finiten Elemente auf die Strömungsprobleme*. Traupel-Festschrift, Juris-Verlag, Zürich, 1974, pp. 143-170.

12. Jakubowski, M.: Perfect Fluid Flow in Turbomachinery of Toroidal Geometry. Ph. D. Dissertation, Northwestern University, Evanston, Illinois, 1965.
University Microfilms, Inc., Ann Arbor, Michigan. (66-2716).
Also published in the article: Jakubowski, M., Kovitz, A. A., and Raynor, S.: Ideal Fluid Flow in an Enclosure of Toroidal Geometry. *Int. J. Non-Linear Mechanics*, Vol. 6, 1971, pp. 101-115.
13. Nilsson, J. E. V.: On the Ideal Flow Through Axial Turbomachine Cascades. Diss. Royal Institute of Technology, Stockholm, 1962.
14. Novak, R. A.: Streamline Curvature Computing Procedures for Fluid-Flow Problems. *Journal of Engineering for Power*, Trans. ASME, Oct. 1967, pp. 478-490.
15. Otte, J. J.: A Method of Analysis of the Axi-Symmetrical Flow in the Blade Channels of Turbines. Third scientific conference on steam turbines of great output, Gdansk, Sept. 24-27, 1974. *Prace instytutu maszyn przeplywowych, Warszawa-Poznan, Zeszyt 70-72*, 1976, pp. 597-614.
16. Robertson, J. M.: Hydrodynamics in Theory and Application. Prentice-Hall, London, 1965.
17. Smith, Jr, L. H.: The Radial-Equilibrium Equation of Turbomachinery. *Journal of Engineering for Power*, Trans. ASME, Jan. 1966. pp. 1-12.
18. Swanson, W. M.: Fluid Mechanics. Holt, Rinehart and Winston Inc. New York, 1970.
19. Szüle, D.: General Relations for the Calculation of the Characteristics of a Hydrodynamic Torque Converter. Proceedings of the fourth conference on fluid machinery, Budapest, 1972, pp. 1397-1408.
20. Traupel, W.: Die Theorie der Strömung durch Radialmaschinen. Verlag G. Braun, Karlsruhe, 1962.
21. Traupel, W.: Thermische Turbomachinen. Springer-Verlag, Berlin, 1966.
22. Wennerstrom, A. J.: On the Treatment of Body Forces in the Radial Equilibrium Equation of Turbomachinery. *Traupel-Festschrift*, Juris-Verlag, Zürich, 1974, pp. 351-367.
23. Weston, E. B.: Theory and Design of Automatic Transmission Components. Butterworths, London, 1967.

24. Whitfield, A., Wallace, F. J., and Sivalingam, R.: A Performance Prediction Procedure for Three Element Torque Converters. *Int. J. Mech. Sci.*, Vol. 20, 1978, pp. 801-814.
25. Wilkinson, D. H.: Stability, Convergence, and Accuracy of Two-Dimensional Streamline Curvature Methods Using Quasi-Orthogonals. *Proc. Instn. Mech. Engrs.* 1969-70, Vol. 184 Pt 3G, pp. 108-119.
26. Wolf, M.: *Strömungskupplungen und Strömungswandler*. Springer-Verlag, Berlin, 1962.
27. Zobory, I.: Theoretical Investigation of the Sphere of Problems Involved with the Inverse Meridian Flow Mode of Action of Hydrodynamical Torque Converters, Based on Computed Characteristic Curves. *Proceedings of the fourth conference on fluid machinery, Budapest, 1972*, pp. 1557-1570.

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